ANALYSIS OF DYNAMICAL PROPERTIES OF DISCRETE AND TIME DELAYED STOCK CONTROL MODELS

by

Ahmet Ö zgül
B.S., Industrial Engineering, Boğaziçi University, 2002

Submitted to the Institute for Graduate Studies in Science and Engineering in partial fulfillment of the requirements for the degree of Master of Science

Graduate Program in Industrial Engineering
Boğaziçi University
2004
ANALYSIS OF DYNAMICAL PROPERTIES OF DISCRETE AND TIME DELAYED STOCK CONTROL MODELS

APPROVED BY:

Prof. Yaman Barlas
(Thesis Supervisor)

Prof. Ali Rana Atilgan

Assoc. Prof. Taner Bilgiç

DATE OF APPROVAL: 12/5/2006
ACKNOWLEDGEMENTS

I would like to express my gratitude to Prof. Yaman Barlas for his guidance and feedback throughout this study.

I would like to thank Prof. Ali Rana Atılgan and Assoc. Prof. Taner Bilgiç for being in my thesis committee to evaluate this work and for their comments and suggestions.
ABSTRACT

ANALYSIS OF DYNAMICAL PROPERTIES OF DISCRETE AND TIME DELAYED STOCK CONTROL MODELS

Four policies frequently used in inventory control management are Order Point - Order Up to Level \((s,S)\) policy, Order Point - Order Quantity \((s,Q)\) policy, Review Period - Order Up to Level \((R,S)\) policy and \((R,s,S)\) policy. Exact properties of inventory dynamics resulting from these policies are not known. Knowledge of these properties would provide important information for inventory managers. In this thesis, dynamics that result from the application of these standard inventory management policies are analyzed under different delay structures: first order continuous delay, \(M^{th}\) order continuous delay and discrete (or mixed) delay structures. All four inventory policies investigated involve nonlinear decision rules, in the form of if-else statements. Since there is not a general mathematical theory to investigate nonlinear systems, innovative methods are used to analyze dynamics of these nonlinear inventory policies and in all cases analytical results are supported with simulation experiments. In the first phase of the thesis, behaviors types (such as goal seeking or periodic) of effective inventory and orders are derived for each policy, as these constitute a foundation for the remaining analysis. In the second phase, inventory dynamics formulas are derived under the assumption of first order continuous delay structure. In the third phase, behavior types of supply line and inventory stocks of each policy and inventory dynamics formulas are derived under the assumption of \(M^{th}\) order continuous delay structure. In the fourth phase, discrete delay structure is analyzed and behavior types of supply line and inventory stocks and inventory dynamics formulas are derived for higher order mixed delay structures. All behavior types and inventory dynamics formulas are summarized in result tables. As a future work, dynamics of inventory may be analyzed under different demand patterns.
ÖZET

KESİK ZAMANLI VE GECİKMELİ STOK DENETİMİ
PROBLEMİNİN DİNAMİK ÖZELLİKLERİNİN İNCELENMESİ

# TABLE OF CONTENTS

ACKNOWLEDGEMENTS .................................................................................. III

ABSTRACT ...................................................................................................... IV

ÖZET .............................................................................................................. V

LIST OF FIGURES ....................................................................................... IX

LIST OF TABLES .......................................................................................... XIV

LIST OF SYMBOLS ....................................................................................... XVIII

1. INTRODUCTION AND LITERATURE REVIEW ...................................... 1

2. PROBLEM DEFINITION AND METHODOLOGY ...................................... 7

3. INVENTORY SYSTEMS AND POLICY DEFINITIONS ............................... 9

   3.1. Order Point - Order Up to Level \((s, S)\) Policy ................................. 15

   3.2. Order Point - Order Quantity \((s, Q)\) Policy ................................. 16

   3.3. Review Period - Order Up to Level \((R, S)\) Policy ....................... 17

   3.4. \((R, s, S)\) Policy ............................................................................. 18

4. DYNAMICS OF EFFECTIVE INVENTORY AND ORDERS ....................... 20

   4.1. Effective Inventory and Orders Dynamics under \((s, S)\) Policy ........ 21

      4.1.1. Mathematical Analysis ............................................................... 21

      4.1.2. Simulation Experiments ........................................................... 24

   4.2. Effective Inventory and Orders Dynamics under \((s, Q)\) Policy ...... 27

      4.2.1. Mathematical Analysis ............................................................... 27

      4.2.2. Simulation Experiments ........................................................... 33

   4.3. Effective Inventory and Orders Dynamics under \((R, S)\) Policy ....... 36

      4.3.1. Mathematical Analysis ............................................................... 36

      4.3.2. Simulation Experiments ........................................................... 39

   4.4. Effective Inventory and Orders Dynamics under \((R, s, S)\) Policy ..... 41

      4.4.1. Mathematical Analysis ............................................................... 41

      4.4.2. Simulation Experiments ........................................................... 42

4.5. Chapter Conclusion ............................................................................. 44

5. INVENTORY DYNAMICS FORMULAS WITH FIRST ORDER DELAYS .... 47

   5.1. Order Point - Order Up to Level \((s, S)\) Policy ................................. 47
5.1.1. Goal Seeking Behavior ................................................. 47
5.1.2. Periodic Behaviors ................................................. 49
5.1.3. Simulation Experiments ............................................ 61

5.2. Order Point - Order Quantity (s, O) Policy ................................................. 64
5.2.1. Goal Seeking Behavior ................................................. 64
5.2.2. Periodic Behaviors ................................................. 66
5.2.3. Simulation Experiments ............................................ 68

5.3. Review Period - Order Up to Level (R, S) Policy ................................................. 70
5.3.1. Goal Seeking Behavior ................................................. 71
5.3.2. Periodic Behaviors ................................................. 71
5.3.3. Simulation Experiments ............................................ 77

5.4. (R, s, S) Policy ................................................................. 79
5.4.1. Goal Seeking Behavior ................................................. 79
5.4.2. Periodic Behaviors ................................................. 80
5.4.3. Simulation Experiments ............................................ 80

5.5. Chapter Conclusion ............................................................. 83

6. INVENTORY DYNAMICS FORMULAS WITH HIGHER ORDER DELAYS .......................... 87
6.1. Analysis of Atomic Structures ............................................ 88
6.1.2. Periodic Inflow-Proportional Outflow Atomic Structure .......... 89
6.1.3. Constant Inflow-Constant Outflow Atomic Structure ............ 92
6.1.4. Periodic Inflow-Constant Outflow Atomic Structure ............ 92

6.2. Atomic Structures in Inventory Policies ............................................. 93
6.2.1. First Supply Line Structure ........................................... 93
6.2.2. Intermediate Supply Line Structures ................................ 94
6.2.3. Inventory Structure ................................................... 95

6.3. Inventory Dynamics of Mth Order Continuous Delay Systems ............ 101

6.4. Simulation Experiments ..................................................... 102
6.5. Chapter Conclusion ............................................................. 109

7. INVENTORY DYNAMICS FORMULAS WITH DISCRETE (MIXED) DELAYS ....................... 112
7.1. Simple Atomic Structures with Discrete Delay ............................... 112
7.1.1. Constant Inflow Atomic Structure .................................... 114
7.1.2. Periodic Inflow Atomic Structure .................................... 116
7.2. Inventory Dynamics of Mixed Delay Systems ........................................... 121
7.3. Simulation Experiments ........................................................................... 123
7.4. Chapter Conclusion .................................................................................. 134

8. CONCLUSION AND FUTURE WORK .......................................................... 135

APPENDIX A: EFFECTIVE INVENTORY DYNAMICS UNDER \((R, s, S)\) POLICY 138

APPENDIX B: DYNAMICS OF INVENTORY UNDER \((R, s, S)\) POLICY .......... 143

APPENDIX C: CONTINUOUS TIME DISCRETE DELAY SYSTEMS ................. 149
  C.1. Equilibrium and Stability Analysis ..................................................... 149
  C.2. Simple Atomic Structures with Discrete Delay .................................. 150
      C.2.1. Constant Inflow Atomic Structure ............................................ 152
      C.2.2. Periodic Inflow Atomic Structure ............................................. 154
  C.3. Inventory Model with Continuous Order Policy .................................. 161

APPENDIX D: SAMPLE MODEL AND EQUATIONS .................................. 166

REFERENCES ....................................................................................... 168
LIST OF FIGURES

Figure 1.1. Generic form of an inventory control problem ............................................. 2
Figure 3.1. Stock flow representation of a first order continuous delay .......................... 9
Figure 3.2. Inflow outflow dynamics of first order continuous delay with $\tau = 6$ ........ 10
Figure 3.3. Stock flow representation of a third order continuous delay ....................... 10
Figure 3.4. Inflow outflow dynamics of 3rd and 1st order continuous delays with delay 6 11
Figure 3.5. Stock flow representation of discrete delay .................................................. 12
Figure 3.6. Inflow outflow dynamics of discrete delay with $\tau = 10$ ............................. 13
Figure 3.7. Stock flow representation of Stock Acquisition sector ............................... 14
Figure 4.1. $(s, S)$ policy, first order continuous delay, goal seeking behavior ............ 25
Figure 4.2. $(s, S)$ policy, second order continuous delay, P-5 oscillation ..................... 26
Figure 4.3. $(s, S)$ policy, third order mixed delay, goal seeking behavior .................... 26
Figure 4.4. $(s, Q)$ policy, decreasing $EI$ and constant order behavior ......................... 33
Figure 4.5. $(s, Q)$ policy, second order continuous delay, goal seeking behavior .......... 34
Figure 4.6. $(s, Q)$ policy, third order continuous delay, P-8 oscillation ....................... 34
Figure 4.7. $(s, Q)$ policy, third order mixed delay, P-5 oscillation ............................. 35
Figure 4.8. $(s, Q)$ policy, first order delay, irrational ratio, P-15 oscillation ................. 36
Figure 4.9. $(R, S)$ policy, first order continuous delay, goal seeking behavior ............. 39
Figure 7.17. \((R,s,S)\) policy, third order mixed delay .................................................. 131

Figure 7.18. \((R,s,S)\) policy, third order mixed delay, phase map of \(I\) and \(SL^2\) ................. 131

Figure C.1. Continuous time, discrete delay one stock atomic structure ............................ 151

Figure C.2. Constant inflow, continuous time, discrete delay system ............................... 153

Figure C.3. Transient variable inflow, continuous time, discrete delay structure .............. 154

Figure C.4. Periodic inflow discrete delay atomic structure, \(\tau = 4\pi\) ................................. 159

Figure C.5. Periodic inflow discrete delay atomic structure, \(\tau = 4\pi, dt = 0.1\) ............ 159

Figure C.6. Periodic inflow discrete delay atomic structure, \(\tau = 5\pi/2\) ............................. 160

Figure C.7. Periodic inflow discrete delay atomic structure, \(\tau = 3\pi/2\) ............................. 160

Figure C.8. Periodic inflow discrete delay atomic structure, \(\tau = 3\pi/2, dt = 0.25\) ....... 161

Figure C.9. Stock flow representation with one goal ......................................................... 161

Figure C.10. Inventory dynamics (\(\tau = \pi\) and \(AT = 3,5,7,9\) for 1\(^{st}\), 2\(^{nd}\), 3\(^{rd}\), and 4\(^{th}\) runs) 164

Figure C.11. Inventory and supply line dynamics \(\tau = \pi\) and \(AT = 2\) ............................. 164

Figure C.12. Inventory dynamics (\(\tau = \pi\) and \(AT = 1.5,1.4,1.3\) for 1\(^{st}\), 2\(^{nd}\) and 3\(^{rd}\) runs). 165

Figure D.1. Stock flow representation of \((s,S)\) policy, 2\(^{nd}\) order continuous delay .... 166
LIST OF TABLES

Table 4.1. \((s,S)\) parameters yielding goal seeking behavior ........................................... 25

Table 4.2. \((s,S)\) parameters yielding P-5 oscillation ...................................................... 25

Table 4.3. \((s,S)\) parameters with mixed delay yielding goal seeking behavior .................... 26

Table 4.4. \((s,Q)\) parameters yielding decreasing \(EI\) and constant order behavior ........... 33

Table 4.5. \((s,Q)\) parameters yielding goal seeking behavior ............................................. 33

Table 4.6. \((s,Q)\) parameters yielding P-8 oscillation ...................................................... 34

Table 4.7. \((s,Q)\) parameters with mixed delay yielding P-5 oscillation .............................. 35

Table 4.8. \((s,Q)\) parameters yielding P-15 oscillation ..................................................... 35

Table 4.9. \((R,S)\) parameters yielding goal seeking behavior ............................................ 39

Table 4.10. \((R,S)\) parameters yielding P-3 oscillation .................................................... 39

Table 4.11. \((R,S)\) parameters with mixed delay yielding P-4 oscillation .......................... 40

Table 4.12. \((R,s,S)\) policy yielding goal seeking behavior ................................................ 42

Table 4.13. \((R,s,S)\) parameters yielding P-3 oscillation .................................................... 43

Table 4.14. \((R,s,S)\) parameters with mixed delay yielding P-4 oscillation ....................... 43

Table 4.15. Summary of effective inventory and orders behavior ....................................... 45

Table 5.1. \((s,S)\) parameters yielding goal seeking behavior ............................................. 48

Table 5.2. \((s,S)\) policy, goal seeking behavior, analytical and simulation results ............... 49
Table 5.3. \((s, S)\) parameters for sequence illustration ............................................ 50

Table 5.4. \((s, S)\) policy, EI dynamics sequence illustration ........................................ 50

Table 5.5. \((s, S)\) parameters yielding P-2 oscillation ................................................ 53

Table 5.6. \((s, S)\) policy, P-2 oscillation, analytical and simulation results .................. 53

Table 5.7. \((s, S)\) parameters yielding P-3 oscillation ................................................ 57

Table 5.8. \((s, S)\) policy, P-3 oscillation, analytical and simulation results .................. 58

Table 5.9. \((s, S)\) parameters yielding P-5 oscillation ................................................ 61

Table 5.10. \((s, S)\) policy, P-5 oscillation, analytical and simulation results .................. 61

Table 5.11. \((s, S)\) parameters yielding P-8 oscillation ................................................ 62

Table 5.12. \((s, S)\) policy, P-8 oscillation, analytical and simulation results .................. 62

Table 5.13. \((s, S)\) parameters yielding P-4 oscillation ................................................ 63

Table 5.14. \((s, S)\) policy, P-4 oscillation, analytical and simulation results .................. 63

Table 5.15. \((s, Q)\) parameters yielding goal seeking SL, ever decreasing I ..................... 65

Table 5.16. \((s, Q)\) parameters yielding goal seeking behavior ...................................... 66

Table 5.17. \((s, Q)\) parameters yielding P-5 oscillation ................................................ 68

Table 5.18. \((s, Q)\) parameters yielding P-4 oscillation ................................................ 69

Table 5.19. \((s, Q)\) parameters yielding P-15 oscillation ............................................. 69

Table 5.20. \((s, Q)\) policy, P-15 oscillation, 2 period simulation results .......................... 70

Table 5.21. \((R, S)\) parameters yielding P-5 oscillation ............................................. 77
Table 6.8. \((R,s,S)\) parameters yielding P-6 oscillation ........................................... 107

Table 6.9. \((R,s,S)\) parameters yielding goal seeking behavior ........................................ 108

Table 6.10. \(SL\) and \(I\) behavior under policies with \(M^{th}\) order continuous delays ........ 110

Table 6.11. \(SL\) and \(I\) formulas under \((s,S),(R,S)\) and \((R,s,S)\) policies .................... 111

Table 7.1. \((s,S)\) parameters with first order discrete delay ......................................... 123

Table 7.2. \((s,S)\) parameters with second order mixed delay .......................................... 124

Table 7.3. \((s,S)\) parameters with third order mixed delay ............................................. 124

Table 7.4. \((s,Q)\) parameters with second order discrete delay ........................................ 125

Table 7.5. \((s,Q)\) parameters with third order mixed delay ............................................ 126

Table 7.6. \((s,Q)\) parameters with third order discrete delay .......................................... 126

Table 7.7. \((R,S)\) parameters with second order discrete delay ........................................ 127

Table 7.8. \((R,S)\) parameters with second order mixed delay .......................................... 128

Table 7.9. \((R,S)\) parameters with third order mixed delay ............................................ 129

Table 7.10. \((R,s,S)\) parameters with second order mixed delay ..................................... 130

Table 7.11. \((R,s,S)\) parameters with third order mixed delay ....................................... 130

Table 7.12. Simulation results third order mixed delay .................................................... 133

Table 8.1. Summary of behavior types of orders ............................................................... 137

Table 8.2. Behavior of different stock types with respect to order behavior ....................... 137
LIST OF SYMBOLS

a
An integer constant

A
A real constant

AT
Adjustment time

b
An integer constant

B
A constant 2 by 2 matrix

D
Positive constant demand

DAVGSL\text{m}
Desired average goods in supply line m

DMINT
Desired minimum inventory

EI_k
Level of effective inventory at time k

f_e
Equilibrium value of receiving rate in the continuous model

f_{k}\text{m}
Outflow rate of supply line stock m between times k and (k + 1)

f(t)
Receiving rate at time t in the continuous model

i
Constant inflow rate

i_k
Inflow rate between times k and (k + 1)

i(t)
Inflow rate at time t in the continuous model

I_e
Equilibrium level of inventory stock

I_k
Level of inventory at time k

I(t)
Level of inventory at time t in continuous model

I'(t)
Rate of change of inventory at time t in continuous model

I^*
Desired inventory level

k
Discrete time step

K
A constant 2 by 1 matrix

M
Order of supply line (delay)

o
Constant outflow rate

o_k
Outflow rate between times k and (k + 1)

o(t)
Outflow rate at time t in the continuous model

O_k
Order amount between times k and (k + 1)
$O(t)$  Order amount at time $t$ in the continuous model

$P$  Period in a periodic behavior

$Q$  Order quantity

$R$  Review period

$s$  Order point

$S$  Upper level of inventory

$SL_e$  Equilibrium level of supply line stock

$SL_k$  Level of supply line stock at time $k$

$SL^m_k$  Level of supply line stock $m$ at time $k$

$SL(t)$  Level of supply line stock at time $t$ in continuous model

$SL'(t)$  Rate of change of supply line at time $t$ in continuous model

$SS$  Safety stocks

$t$  Continuous time

$t_0$  Initial time

$y(t)$  Some unknown variable

$y'(t)$  Rate of change of $y$

$\lambda$  Eigen value

$\mu$  Real constant

$\tau$  Delay constant

$\tau_m$  Delay constant of supply line stock $m$

$\tau_T$  Average total delay

$\phi(t)$  Some initial function
1. INTRODUCTION AND LITERATURE REVIEW

System Dynamics is a methodology for modeling, analyzing and improving dynamical socio-economic and managerial systems, using a feedback perspective. Among many other topics, stock control problem constitutes a typical topic for system dynamics analysis. Stock control problem is a common dynamic decision problem in which decision maker tries to maintain the level of a stock within an acceptable range by controlling its inflows and outflows. Inventory control problem, subject of this thesis, is one of the most common examples of the generic stock control problem [1,2].

Inventory control is a critical aspect of successful management. With high varying costs, companies can not afford to have money tied up in excess inventories. The objective of good customer service and efficient production must be met at minimum inventory levels [3].

The well-known Beer Production-Distribution Inventory model represents a supply chain, in which there are four levels; factory, distributor, wholesaler and retailer. In System Dynamics literature it is shown that the Beer Production-Distribution Inventory model may show a great variety of complex behaviors depending upon the assumed ordering policy [4]. Parameter values estimated from real-life decision makers’ data may even cause chaotic behavior in Beer Distribution Game [5]. However, these two papers are semi analytical and visual. In this thesis, we concentrate on a single level (i.e. retailer) and present a complete mathematical analysis of the dynamics of inventory under \((s,S)\), \((s,Q)\), \((R,S)\) and \((R,s,S)\) policies which are explained below. In our model, we assume a constant positive demand (of a single product) and wholesaler (the previous level in the supply chain) is of infinite capacity (all orders are satisfied).

Generic form of an inventory control problem is shown in Figure 1.1. In the Goal Formation sector, policy parameters are calculated (such as order point \(s\)). Outcome of the Decision/Policy Rule sector is an order decision (such as order = \(Q\)). However, this order decision does not affect the inventory directly. In real life there are time lags between initiation of control actions and their effects. These time lags are formulated by delay
structures in System Dynamics models. Delay structures are formed by using supply line stocks which accumulate the order decisions that have not yet reached the inventory due to time lags. Delay structures (supply line stocks) together with inventory stock form the Stock Acquisition sector which represents the physical flow of materials.

![Diagram showing the generic form of an inventory control problem](image)

Figure 1.1. Generic form of an inventory control problem

Time lags in real life are modelled by two different forms of delay structures in System Dynamics models; continuous and discrete. Continuous delays are further divided into first order and higher order continuous delays. The delay structure in a particular situation may be modelled by a first order continuous delay, higher order continuous delay, discrete delay or combination of discrete and continuous delays. The important point is that in discrete delays all input material is delivered after some delay period while in continuous delays the delivery is expanded over the delay interval. If the supplier delivers all the order at once, discrete delay structures are employed. For some reasons supplier may not be able to deliver all the order at once, in which case it is more logical to use continuous delay structures. There may be different interpretations of delay structures. In one case, each delay structure may model a different entity between the placement of an order and its receipt. In another case, the modeller does not want to model the details but knows the general pattern of how the orders come in which higher order continuous delays may be more appropriate to model these kinds of time lags.
Delays are inherent in many physical and engineering systems. Any kind of system where substances, information or energy is being transmitted to certain distances, experiences time lags due to transportation time. An additional time-lag might arise due to the time needed for certain measurements to be taken or for the system to sense information and react on it [6]. In many applications, it is assumed that the future state of the system is independent of the past and depends only on the present. In many real life phenomena direct causality assumption (future state depends only on present) is only an approximation of true situation and a more realistic model would also include past values as well as the present values of the system [7]. Therefore, it is realistic to analyze dynamics of these inventory control policies under discrete delay structures. Including discrete delays makes the system a set of nonlinear delay-difference equations, mathematically rather difficult to analyze.

One frequently used inventory decision rule in System Dynamics literature is "anchor and adjust" rule in which a starting point (called the anchor) is identified and necessary adjustments are made [1]. Analysis of dynamical properties of anchor and adjust rule is almost complete [2]. Order Point - Order Up to Level \((s,S)\) policy, Order Point - Order Quantity \((s,Q)\) policy, Review Period - Order Up to Level \((R,S)\) policy and \((R,s,S)\) policy are four policies often used in inventory control management. Exact properties of the dynamics resulting from these policies are not known. \((s,S)\) and \((s,Q)\) policies are continuous review policies in which "how many" and "when" to order information is specified. \((R,S)\) and \((R,s,S)\) policies are periodic review policies in which "how many" and "how often" to order information is specified [8].

All models in this thesis are discrete time systems (except those in Appendix C). Since \((s,S)\) and \((s,Q)\) policies are called continuous review policies, there may seem to be a contradiction that must be explained. Firstly, \((s,S)\) and \((s,Q)\) models are not truly continuous in the mathematical sense, because inventory models are typically discrete in time. The term "continuous" is used to mean that there is no "periodic review", that the inventory can be reviewed at "sufficiently" small time steps. Therefore, if we change the interpretation of discrete time from a larger time interval to a smaller interval (for example from week to day, to hour), our discrete time models converge to continuous models,
although still not truly continuous. If we specify discrete time step as a small enough step, we define “continuous” as “frequent enough” rather than truly continuous in the limit.

In general, nonlinear systems have the ability to show rich behavior patterns and there is not a general theory to analyze nonlinear systems. Since decision rules in the standard inventory control policies are nonlinear, they may show complex behavior.

Dynamics that result from the above inventory policies are not known and the objective of this thesis is to derive dynamical properties of these inventory policies under some general assumptions. In this thesis, we are not concerned with finding the optimum policy parameters with respect to some given cost structure. We assume that some optimal or satisfactory policy parameters are already provided.

In dynamical systems (such as inventory systems), optimal policy (parameters) and solution properties provide two different types of information. When optimal policy is known in a particular situation, knowledge of dynamics (solutions) of main variables would be further important information to managers. Especially in dynamical systems, it is important to question optimal policies before implementing them. In the following three paragraphs we discuss three general implementation questions to which our results provide answers.

The first implementation question refers to the case of more than one optimal policy. According to the underlying assumptions, there may be alternative optimal policies. At such a point, the question is which one to choose. If assumptions of all optimal policies are sound, a manager is indifferent between those policies. However, if managers know the corresponding behavior types resulting from these optimal policies, or stability properties, indicators of amplitudes of oscillations or in the best case exact points of dynamics of inventory, they can easily choose that policy with the least potential implementation problems (such as the one making less fluctuations in inventories).

The second implementation question has to do with sudden uncertainty situations. In this case, we assume there exists an optimal policy whose assumptions are sound. In real life, there may be sudden uncertainty situations which can not be taken into consideration
while finding optimal policies. One example of such an uncertainty situation is sudden high demand from an important customer for relatively short time periods. In such a case, if the future level of inventory is known beforehand, managers will be able to take better precautions to meet that demand.

In this third case, we assume an optimal policy whose assumptions are sound and no sudden uncertainty. Even in such cases, knowledge of exact points of inventory dynamics may enable managers to make better planning and increase their bargaining power in many situations. For example, we assume a rental space is used to store inventory and the time step is rather a long period. In such a case, if it is known that inventory displays periodic oscillation with high levels for some months and low levels in months to follow, managers can make better planning and bargain alternative rental payment plans which can not be done if this information is unknown.

This thesis fills a gap between two different groups; system dynamics and optimization circles. System Dynamics literature has its own ordering rules (such as anchor and adjust) and is not generally interested in the optimal inventory policies we investigate in this thesis which are used in inventory control literature. Optimization, on the other hand, is not interested in dynamics of these policies. Therefore, in this thesis we combine these two different views.

We first investigate behavior of effective inventory and orders of retailer under different policies. These results are fundamental for all the remaining analysis since they are valid for any order of supply line and any delay structure. We specify order of delay as one and delay type to continuous and investigate inventory dynamics of retailer. We derive inventory and supply line value formulas under first order continuous delay structure. Secondly, we relax the first constraint, increase order of delay but still keeping the delay type continuous. We investigate inventory dynamics of retailer under higher order continuous delays. We derive inventory dynamics formulas and present behavior types of supply line stocks and inventory. Lastly, we also relax the second constraint, for higher order delays where delay types are mixed (both continuous and discrete are used to model time lags). We investigate inventory dynamics under higher order mixed delays. We derive inventory dynamics formulas and present behavior types of supply line and inventory.
In Chapter 2, research topic is defined and methodology used is stated. In Chapter 3, definitions of each inventory policy are given. In Chapter 4, dynamics of the effective inventory (inventory position) and orders are analyzed. In Chapters 5, 6 and 7, inventory dynamics with first order continuous delay, $M^{th}$ order continuous delay and discrete (or mixed) delays are analyzed respectively. In Chapter 8, conclusions and future work are stated.
2. PROBLEM DEFINITION AND METHODOLOGY

There is no general reported analysis of the dynamics of inventories that result from the application of the standard inventory management policies \((s,S)\), \((s,Q)\), \((R,S)\) and \((R,s,S)\). The purpose of this thesis is to derive the dynamical consequences of these nonlinear inventory policies under some general assumptions.

Order Point - Order Up to Level \((s,S)\) and Order Point - Order Quantity \((s,Q)\) policies are "continuous review" policies. Whenever effective inventory (inventory position) drops to the order point \(s\) or lower, effective inventory is increased to order up to level \(S\) in \((s,S)\) policy while a fixed quantity \(Q\) is ordered in \((s,Q)\) policy. Review Period - Order Up to Level \((R,S)\) and \((R,s,S)\) policies are "periodic review" policies. In \((R,S)\) policy, effective inventory is increased to the order up to level \(S\) every \(R\) units of time. \((R,s,S)\) policy is a combination of \((s,S)\) and \((R,S)\) policies. In this policy, effective inventory is checked at every \(R\) units of time. If it is at or below order point \(s\), effective inventory is increased to the order up to level \(S\). If it is above order point \(s\), no action is taken until the next review period [8].

All inventory policies stated above involve nonlinear decision rules, in the form of if-else statements. Nonlinear systems have the ability to show rich behavior patterns. Since there is not a general mathematical theory to analyze nonlinear systems, innovative methods are used to analyze dynamics of these nonlinear inventory policies.

In the well-known Beer Production-Distribution Inventory model, we concentrate on a single level (i.e. retailer) and present a complete mathematical analysis of the dynamics of inventory under the four inventory policies stated above. In our models, we assume a constant positive demand (of a single product) and supplier (the previous level in the supply chain) is of infinite capacity (all orders are satisfied). We are not concerned with finding the optimum policy parameters with respect to some given cost structure. We assume that some optimal or satisfactory policy parameters are already provided.
This thesis fills a gap between two different groups; system dynamics and optimization circles. While System Dynamics literature has its own ordering rules (such as anchor and adjust) and is not generally interested in these optimal inventory policies, optimization circles are not interested in dynamics of these policies. Therefore, in this thesis we combine these two different views.

Dynamics that result from the application of the standard inventory management policies \((s, S), (s, Q), (R, S)\) and \((R, s, S)\) are analyzed under different delay structures: first order continuous delay, \(M^{th}\) order continuous delay and discrete (or mixed) delay structures.

The first sub-objective of this research is to derive properties of inventory dynamics under first order continuous delay structure. The second sub-objective is to analyze dynamical properties of these inventory control policies under higher order continuous delay structures, since it is known from System Dynamics literature that as the number of stock variables increases to three or above in the model, the system may show complex behaviors. Finally, in many real life situations continuous delay structures do not represent the system appropriately, in which case discrete delay structures are employed. To analyze dynamical properties of these nonlinear inventory control policies under discrete and mixed delay structures forms the third sub-objective of this research. With discrete delays, the system becomes a nonlinear delay difference equation model, mathematically difficult to analyze.

Since the inventory control models are discrete time systems, difference dynamics is adopted; equilibrium analysis, stability techniques and periodicity results of difference dynamics are used. In all cases, analytical techniques and proofs are supported by simulation experiments.
3. INVENTORY SYSTEMS AND POLICY DEFINITIONS

In this chapter, the inventory policies are defined by specifying the three different sections of the generic inventory control problem shown in Figure 1.1.

In any system there exists time lags between initiation of control actions and their effects, therefore order decisions taken in inventory systems do not affect the inventory directly. In System Dynamics models these time lags are represented by supply line stocks (delay structures).

There exist two possible forms of delay structures; continuous and discrete. The first delay structure is continuous delay. Stock flow representation of a first order continuous delay is given in Figure 3.1.

![Diagram of a first order continuous delay](image)

Figure 3.1. Stock flow representation of a first order continuous delay

First order continuous delay outflow is defined by:

\[ o_k = SL_k / \tau \]

where \( SL_k \) (Supply line) is the level of supply line stock at time \( k \), \( o_k \) (out) is the outflow rate between times \( k \) and \( (k + 1) \) and \( \tau \) (delay) is delay.

The supply line stock is defined by:
$SL_k = SL_{k-1} + i_{k-1} - o_{k-1}$

where $i_k$ (in) is the inflow rate between times $k$ and $(k + 1)$.

Inflow and outflow dynamics of first order continuous delay are shown in Figure 3.2.

![Figure 3.2. Inflow outflow dynamics of first order continuous delay with $\tau = 6$](image)

Higher order delays are obtained by serially connecting first order delay structures, such that the output of a delay serves as the input of the following delay structure. As an example, stock flow representation of a third order continuous delay is given in Figure 3.3 (assuming that the system has total average delay $\tau_T$).

![Figure 3.3. Stock flow representation of a third order continuous delay](image)

Third order continuous delay outflows are defined by:
\[ f_k^1 = \frac{SL_k^1}{\tau_1} \]
\[ f_k^2 = \frac{SL_k^2}{\tau_2} \]
\[ o_k = \frac{SL_k^3}{\tau_3} \]

where \( SL_k^m \) is the level of supply line stock \( m \) at time \( k \), \( f_k^m \) (flow \( m \)) is the outflow of supply line \( m \) between times \( k \) and \( (k+1) \), \( o_k \) is the outflow rate between times \( k \) and \( (k+1) \). \( \tau_T \) is the total average delay and is calculated as:

\[ \tau_T = \tau_1 + \tau_2 + \tau_3 \]

where \( \tau_m \) is the delay of supply line stock \( m \).

The supply line stocks are defined by:

\[ SL_k^1 = SL_{k-1}^1 + i_{k-1} - f_{k-1}^1 \]
\[ SL_k^2 = SL_{k-1}^2 + f_{k-1}^1 - f_{k-1}^2 \]
\[ SL_k^3 = SL_{k-1}^3 + f_{k-1}^2 - o_{k-1} \]

where \( i_k \) is the inflow rate between times \( k \) and \( (k+1) \).

Figure 3.4. Inflow outflow dynamics of 3rd and 1st order continuous delays with delay 6
Inflow and outflow dynamics of a third order continuous delay compared with first order continuous delay are shown in Figure 3.4 (The total average delay $\tau_T$ in third order continuous delay and the delay $\tau$ of first order continuous delay are set to be same at 6).

Second delay structure is discrete delay. Stock flow representation of discrete delay is given in Figure 3.5.

![Diagram of stock flow representation of discrete delay](image)

Figure 3.5. Stock flow representation of discrete delay

Discrete delay outflow is defined by:

$$o_k = i_{k-\tau}$$

where $o_k$ and $i_k$ are outflow and inflow rates between times $k$ and $(k + 1)$ respectively and $\tau$ is delay.

The supply line stock is defined by:

$$SL_k = SL_{k-1} + i_{k-1} - o_{k-1}$$

where $SL_k$ is the level of supply line stock at time $k$.

Inflow and outflow dynamics of discrete delay stock are shown in Figure 3.6.
When Figure 3.4 and Figure 3.6 are compared, a behavioral difference between continuous and discrete delay structures is observed; in continuous delays outflow has a distribution around average value where as in discrete delays all input quantity is delivered at once after some delay period $\tau$.

The time lag in a real life case may be modelled by a first order continuous delay, higher order continuous delay, discrete delay or combination of discrete and continuous delays. The important point is that in discrete delays all input material is delivered after some delay period while in continuous delays the delivery is expanded over the delay interval. If a supplier delivers all the order at once, discrete delay structures are employed. For some reasons the supplier may not be able to deliver all the order at once, in which case it is more logical to use continuous delay structures.

There may be different interpretations of delay structures. In one case, each delay structure may model a different entity between the placement of an order and its receipt. In another case, the modeller does not want to model the details but knows the general pattern of how the orders come in which case higher order continuous delays may be more appropriate to model these kinds of time lags (for example orders arrive less at the beginning, then an increase occurs in the arrival of orders and lastly a decrease again). Lastly, if time window (numerical time step) is largened, continuous delays are more appropriate than discrete delays (for example if numerical time step is one day it is logical
to represent arrival of orders by discrete delays, however if it is largened to one month discrete delay will not be a correct representation of arrival of orders).

Supply line stocks together with inventory stock form the Stock Acquisition sector which represents the physical flow of materials. Since physical flow of materials does not depend on policies, this sector is represented by the same set of equations for all inventory policies. As an example, Figure 3.7 illustrates the stock flow representation of Stock Acquisition sector under second order continuous delay.

![Stock Acquisition System Diagram]

Figure 3.7. Stock flow representation of Stock Acquisition sector

Behavior of first supply line stock is defined by:

\[ SL^1_k = SL^1_{k-1} + O_{k-1} - f^1_{k-1} \]

where \( SL^1_k \) is the level of first supply line stock at time \( k \), \( f^1_k \) is the outflow of first supply line stock between times \( k \) and \((k+1)\) and \( O_k \) (order) is the order decision between times \( k \) and \((k+1)\).

Behavior of any intermediate supply line stock \( m \) is defined by:

\[ SL^m_k = SL^m_{k-1} + f^{m-1}_{k-1} - f^m_{k-1}, \quad m = 2, 3, ..., M \]
where $SL_m^k$ is the level of supply line stock $m$ at time $k$ and $f_m^k$ is the outflow of supply line stock $m$ between times $k$ and $(k+1)$.

Behavior of inventory is defined by:

$$I_k = I_{k-1} + f_{k-1}^M - D$$

where $I_k$ (Inventory) is the level of inventory at time $k$ and $D$ (demand) is demand.

As a result, for the most general case, where order of delay (supply line) is $M$, delay type and flow equations are not specified, Stock Acquisition sector (physical flow of goods) is defined by the following equation set, for all inventory policies.

$$I_k = I_{k-1} + f_{k-1}^M - D$$
$$SL_m^k = SL_{k-1}^m + f_{k-1}^{m-1} - f_{k-1}^m, \quad m = 2, \ldots, M$$
$$SL_i^k = SL_{k-1}^i + O_{k-1} - f_{k-1}^i$$

Referring back to Figure 1.1, Goal Formation and Decision/Policy Rule sectors differ for each inventory policy. In Goal Formation sector policy parameters are calculated and in Decision/Policy Rule sector order decision is made; these sectors are explained in the following sections of this chapter.

### 3.1. Order Point - Order Up to Level $(s, S)$ Policy

Order Point-Order Up to Level rule is mathematically expressed as:

$$O_k = S - EI_k \quad \text{if } (EI_k \leq s)$$
$$0 \quad \text{else}$$

where $EI_k$ represents effective inventory (inventory position), $s$ the order point and $S$ the upper level of inventory. Effective inventory is calculated as:
$$EI_k = I_k + \sum_{m=1}^{M} SL_k^m$$

where $I_k$ and $SL_k^m$ represent goods in inventory and supply line $m$ respectively.

Order point $s$ and upper level of inventory $S$ can be defined as follows:

$$s = \sum_{m=1}^{M} DAVGSL^m + DMINT$$

$$S = s + Q$$

where $DAVGSL^m$ refers to the desired average goods in supply line $m$ and $DMINT$ to desired minimum inventory. Desired average goods in supply line $m$ and desired minimum inventory can be defined as follows:

$$DAVGSL^m = \tau_m D$$

$$DMINT = D + SS$$

where $\tau_m$ is the receiving delay of supply line stock $m$, $D$ is demand and $SS$ is safety stock. Order quantity $Q$ is determined as constant and safety stock $SS$ is taken as $0$ without loss of generality (safety stock is not necessary when demand is deterministic) for all simulation experiments in the following chapters.

3.2. Order Point - Order Quantity $(s, Q)$ Policy

Order Point-Order Quantity rule is mathematically expressed as:

$$O_k = Q \quad \text{if } (EI_k \leq s)$$

$$0 \quad \text{else}$$

where $EI_k$ represents the effective inventory and $s$ the order point. Effective inventory is calculated as:
$$EI_k = I_k + \sum_{m=1}^{M} SL_k^m$$

where $I_k$ and $SL_k^m$ represent goods in inventory and supply line $m$ respectively.

Order point $s$ can be defined as follows:

$$s = \sum_{m=1}^{M} DAVGSL^m + DMINT$$

where $DAVGSL^m$ refers to the desired average goods in supply line $m$ and $DMINT$ to desired minimum inventory. Desired average goods in supply line $m$ and desired minimum inventory can be defined as follows:

$$DAVGSL^m = \tau_m D$$
$$DMINT = D + SS$$

where $\tau_m$ is the receiving delay of supply line stock $m$, $D$ is demand and $SS$ is safety stock. Order quantity $Q$ is determined as constant and safety stock $SS$ is taken as 0 without loss of generality (safety stock is not necessary when demand is deterministic) for all simulation experiments in the following chapters.

3.3. Review Period - Order Up to Level $(R, S)$ Policy

Review Period-Order Up to Level rule is mathematically expressed as:

$$O_k = S - EI_k$$
$$0$$

if $(\text{mod}(k, R) = 0)$
else

where $EI_k$ represents the effective inventory, $S$ the upper level of inventory and $R$ is the review period. Effective inventory is calculated as:
\[ EI_k = I_k + \sum_{m=1}^{M} SL_k^m \]

where \( I_k \) and \( SL_k^m \) represent goods in inventory and supply line \( m \) respectively.

Upper level of inventory \( S \) can be defined as follows:

\[ S = \sum_{m=1}^{M} DAVGSL^m + DMINT + RD \]

where \( DAVGSL^m \) refers to the desired average goods in supply line \( m \) and \( DMINT \) to desired minimum inventory. Desired average goods in supply line \( m \) can be defined as follows:

\[ DAVGSL^m = \tau_m D \]

where \( \tau_m \) is the receiving delay of supply line stock \( m \) and \( D \) is demand. Desired minimum inventory \( DMINT \) corresponds to safety stocks. It is arbitrarily fixed as \( D \) for all simulation experiments in the following chapters.

3.4. \((R,s,S)\) Policy

\((R,s,S)\) policy is mathematically expressed as:

\[ O_k = S - EI_k \quad \text{if} \quad (\mod(k,R) = 0 \quad \text{and} \quad EI_k \leq s) \]
\[ 0 \quad \text{else} \]

where \( EI_k \) represents the effective inventory, \( s \) the order point, \( S \) the upper level of inventory and \( R \) is the review period. Effective inventory is calculated as:

\[ EI_k = I_k + \sum_{m=1}^{M} SL_k^m \]
where $I_k$ and $SL_k^m$ represent goods in inventory and supply line $m$ respectively.

Order point $s$ and upper level of inventory $S$ can be defined as follows:

$$SS = DMINT + RD$$
$$s = \sum_{m=1}^{M} DAVGSL^m + SS$$
$$S = s + RD$$

where $DAVGSL^m$ refers to the desired average goods in supply line $m$ and $DMINT$ to desired minimum inventory. Desired average goods in supply line $m$ can be defined as follows:

$$DAVGSL^m = \tau_m D$$

where $\tau_m$ is the receiving delay of supply line stock $m$ and $D$ is demand. $DMINT$ is arbitrarily fixed as $D$ for all simulation experiments in the following chapters.

As an example, stock flow diagram and equations of $(s, S)$ policy with second order continuous delay structure are given in Appendix D.
4. DYNAMICS OF EFFECTIVE INVENTORY AND ORDERS

In this chapter, dynamics of effective inventory (inventory position) and order for each inventory policy are analyzed since they constitute a foundation for the remaining analysis.

Effective inventory is sum of inventory and supply line stock(s) at any time point $k$.

$$EI_k = I_k + \sum_{m=1}^{M} SL_k^m$$ (4.1)

As stated in Chapter 3, Stock Acquisition sector (physical flows of goods) does not depend on policies. Therefore it is represented by the following equation set for all inventory polices, where order of supply line is $M$ and delay type is not specified.

$$I_k = I_{k-1} + f_{k-1}^M - D$$
$$SL_k^m = SL_{k-1}^m + f_{k-1}^{m-1} - f_{k-1}^m, \quad m = 2, \ldots, M$$
$$SL_k^1 = SL_{k-1}^1 + O_{k-1} - f_{k-1}^1$$

If definition of effective inventory (Equation (4.1)) is considered together with Stock Acquisition sector equations, the following relationship is obtained for effective inventory.

$$EI_k = EI_{k-1} + O_{k-1} - D$$ (4.2)

Neither order of supply information nor delay type information is seen in Equation (4.2). In terms of effective inventory, flow within supply line stock(s) and between inventory is unimportant (delay types and flow equations are unimportant); order is the only external inflow, demand is the only outflow. Thus (order-demand) is the net change at any time point. Therefore order of supply line and delay type (continuous or discrete) do not affect the analysis. In fact, that is why this chapter constitutes a foundation. In the following sections, order of supply line is taken as $M$ and delay type is not specified.
4.1. Effective Inventory and Orders Dynamics under \((s, S)\) Policy

Order Point - Order Up to Level rule is mathematically expressed as:

\[
O_k = S - EI_k \quad \text{if} \quad (EI_k \leq s) \quad \text{if} \quad (EI_k \leq s) \\
0 \quad \text{else}
\]  \hspace{1cm} (4.3)

From Equations (4.2) and (4.3):

If \( EI_{k-1} \leq s \Rightarrow EI_k = S - D \quad \text{if} \quad (EI_k \leq s) \)

If \( EI_{k-1} > s \Rightarrow EI_k = EI_{k-1} - D \quad \text{if} \quad (EI_k > s) \)  \hspace{1cm} (4.5)

4.1.1. Mathematical Analysis

**Proposition:** \( EI \) can not indefinitely remain greater than order point \( s \). If \( EI > s \Rightarrow O = 0 \) from Equation (4.3) and \( D > 0 \). Therefore once effective inventory is above order point \( s \), it starts to decrease and it certainly drops to order point \( s \) or lower in a finite number of time steps; i.e. at some time point \((k - 1)\), \( EI_{k-1} \leq s \) is certainly satisfied.

**Proposition:** \( EI \) exhibits goal seeking behavior and stays constant at \( S - D \) if \((S - s) \leq D \) (or \((S - D) \leq s\)). At some time point \((k - 1)\), \( EI_{k-1} \leq s \) is certainly satisfied. From Equation (4.4) if \( EI_{k-1} \leq s \Rightarrow EI_k = S - D \leq s \), again from Equation (4.4) if \( EI_k \leq s \Rightarrow EI_{k+1} = S - D \leq s \).

If effective inventory stays constant at \( S - D \), the order \( O_k \) stays constant at \( D \) from Equation (4.3).

**Proposition:** \( EI \) can not indefinitely remain on one side of order point \( s \) if \((S - s) > D \) (or \((S - D) > s\)). At some time point \((k - 1)\), \( EI_{k-1} \leq s \) is certainly satisfied. From Equation (4.4), if \( EI_{k-1} \leq s \Rightarrow EI_k = S - D > s \). An important sub result is that after
EI drops to \( s \) or lower, there must be at least one time point where EI becomes greater than \( s \).

**Proposition:** If \((S-s) > D\), EI displays period-\( n \) oscillation where \( n \) is the smallest integer greater than or equal to the ratio \((S-s)/D\) and it drops to order point \( s \) or lower exactly once in any period. Assume the following alternating behavior sequence for effective inventory.

\[
\begin{align*}
EI_{k-1} &\leq s \\
EI_j > s, j = k, k+1, \cdots, k+n-2 \\
EI_{k+n-1} &\leq s \\
EI_j > s, j = k+n, k+n+1, \cdots, k+n+l-2 \\
EI_{k+n+l-1} &\leq s
\end{align*}
\]  

(4.6)

There is no loss of generality in the proposed alternating behavior sequence. Going below and above \( s \) is not a restriction since it is proven that if \((S-s) > D\), EI can not remain indefinitely on one side of order point \( s \). Most importantly it is not assumed that this behavior sequence indefinitely repeats itself or is complete (it is just any portion of effective inventory behavior of length \( n+l+1 \)). Using Equations (4.4), (4.5) and (4.6):

\[
\begin{align*}
EI_{k-1} &\leq s \Rightarrow EI_k = S-D \\
\{EI_k > s \Rightarrow EI_{k+1} = EI_k - D = S-2D \\
\vdots \\
EI_{k+n-1} > s \Rightarrow EI_{k+n-2} = EI_{k+n-3} - D = S-(n-1)D \\
EI_{k+n-2} > s \Rightarrow EI_{k+n-1} = EI_{k+n-2} - D = S-nD \\
EI_{k+n-1} &\leq s \Rightarrow EI_{k+n} = S-D \\
\{EI_{k+n} > s \Rightarrow EI_{k+n+1} = EI_{k+n} - D = S-2D \\
\vdots \\
\{EI_{k+n+l-3} > s \Rightarrow EI_{k+n+l-2} = EI_{k+n+l-3} - D = S-(l-1)D \\
EI_{k+n+l-2} > s \Rightarrow EI_{k+n+l-1} = EI_{k+n+l-2} - D = S-ID \\
EI_{k+n+l-1} &\leq s
\end{align*}
\]

\[
\begin{align*}
\{EI_k > s \Rightarrow S-D > s \\
EI_{k+1} > s \Rightarrow S-2D > s \\
\vdots \\
EI_{k+n-1} > s \Rightarrow S-(n-1)D > s \\
\{EI_{k+n} > s \Rightarrow S-nD \leq s \\
EI_{k+n+1} > s \Rightarrow S-2D > s \\
\vdots \\
\{EI_{k+n+l-2} > s \Rightarrow S-(l-1)D > s \\
EI_{k+n+l-1} &\leq s \Rightarrow S-ID \leq s
\end{align*}
\]
where effective inventory values in the same symbol have the same position with respect to order point $s$; either greater than $s$ or less than or equal to $s$.

It can be easily shown that the following 4 equations can not be satisfied simultaneously unless $l$ is equal to $n$.

\begin{align*}
EI_{k+n-2} &= S - (n-1)D > s \\
EI_{k+n-3} &= S - (l-1)D > s \\
EI_{k+n-1} &= S - nD \leq s \\
EI_{k+n+l-1} &= S - lD \leq s
\end{align*}

(4.7)

Assume $l$ is not equal to $n$, there are 2 possibilities.

Either $l < n \Rightarrow l \leq n - 1 \Rightarrow S - lD \geq S - (n - 1)D$. Equation (4.8) clearly contradicts this inequality.

\begin{align*}
EI_{k+n-2} &= S - (n-1)D > s \quad \text{and} \quad EI_{k+n+l-1} = S - lD \leq s
\end{align*}

(4.8)

Or $n < l \Rightarrow n \leq l - 1 \Rightarrow S - nD \geq S - (l - 1)D$. Equation (4.9) clearly contradicts this inequality.

\begin{align*}
EI_{k+n-1} &= S - nD \leq s \quad \text{and} \quad EI_{k+n+l-2} = S - (l-1)D > s
\end{align*}

(4.9)

As a result, it is shown that if $l$ is not equal to $n$, Equation (4.7) is a self contradictory set, therefore $l$ must be equal to $n$. $l = n$ means number of time points where $EI$ is greater than $s$ between two time points where $EI$ is less than or equal to $s$ must be constant. So, even if we continue to write the behavior sequence of effective inventory (remember it is not asserted that it is complete), we must obey the upper result. That is, $EI$ displays periodic oscillation in which at only one time point it drops to $s$ or lower.

In fact, this result is very intuitive. Whenever $EI$ drops to $s$ or lower, from Equation (4.4) the next time point it goes the value $S - D$. Then until it decreases down to $s$ or lower, the order $O_k$ is zero from Equation (4.3), and the demand $D$ is a positive constant.
If you start from the same constant $S - D$ and decrease by a positive constant $D$, you certainly pass from the same points and reach the same first number which is lower than or equal to order point $s$.

By definition, period is the length from (including) a time point where effective inventory is less than or equal to $s$ until the next time point where it again drops to $s$ or lower. In Equation (4.6), at time $(k - 1)$ $EI_{k-1} \leq s$ and again at time $(k + n - 1)$ $EI_{k+n-1} \leq s$, therefore effective inventory displays period-$n$ oscillation where $n$ is given by the following equation.

$$EI_{k+n-1} = S - nD \leq s \Rightarrow (S - s)/D \leq n$$

Thus, period of $EI$ is the smallest integer greater than or equal to the ratio $(S - s)/D$.

If effective inventory displays period-$n$ oscillation so does the order $O_k$ from Equation (4.3). In fact we know more than this about orders' behavior. Since effective inventory drops to $s$ or lower once in a period, order decision is made once (at this point the order quantity is $S - EI = S - (S - nD) = nD$) in a period and at other points in that period the order $O_k$ is zero. Thus, order displays period-$n$ oscillation with points $(nD, 0, \ldots, 0)$ where number of zeros is $n - 1$ (period length minus 1).

To summarize, both effective inventory and order exhibit goal seeking behavior if $(S - s) \leq D$ and they both display period-$n$ oscillation if $(S - s) > D$ where $n$ is the smallest integer greater than or equal to ratio $(S - s)/D$.

4.1.2. Simulation Experiments

Otherwise stated, in all experiments of this chapter, delays are continuous.
Example 1: Consider parameters in Table 4.1. They satisfy \((S - s) = 3 \leq D = 5\) therefore goal seeking behavior must result as confirmed in Figure 4.1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>D</th>
<th>Q</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>S</th>
<th>(t_0)</th>
<th>(SL_0)</th>
<th>S-D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>15</td>
<td>18</td>
<td>0</td>
<td>0</td>
<td>13</td>
</tr>
</tbody>
</table>

Figure 4.1. \((s, S')\) policy, first order continuous delay, goal seeking behavior

Example 2: Consider parameters in Table 4.2. They satisfy \((S - s)/D = 18/4 \leq n\) therefore period-5 oscillation must result as confirmed in Figure 4.2.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>D</th>
<th>Q</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>S</th>
<th>(t_0)</th>
<th>(SL_0)</th>
<th>(SL'_0)</th>
<th>S-D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>2</td>
<td>4</td>
<td>18</td>
<td>5</td>
<td>2</td>
<td>32</td>
<td>50</td>
<td>0</td>
<td>0</td>
<td>46</td>
</tr>
</tbody>
</table>


Figure 4.2. \((s, S)\) policy, second order continuous delay, \(P - 5\) oscillation

Example 3: Consider parameters in Table 4.3. They satisfy \((S - s) = 1 \leq D = 2\) so goal seeking behavior must result as confirmed in Figure 4.3. In this example, first and third delay structures are discrete while second one is continuous.

Table 4.3. \((s, S)\) parameters with mixed delay yielding goal seeking behavior

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>D</th>
<th>Q</th>
<th>(t_1)</th>
<th>(t_2)</th>
<th>(t_3)</th>
<th>s</th>
<th>I_0</th>
<th>SL'</th>
<th>SL'</th>
<th>SL'</th>
<th>S-D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>14</td>
<td>15</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>13</td>
</tr>
</tbody>
</table>

Figure 4.3. \((s, S)\) policy, third order mixed delay, goal seeking behavior
4.2. Effective Inventory and Orders Dynamics under \((s,Q)\) Policy

Order Point - Order Quantity rule is mathematically expressed as:

\[
O_k = \begin{cases} 
Q & \text{if } (EI_k \leq s) \\
0 & \text{else}
\end{cases} \quad (4.10)
\]

From Equations (4.2) and (4.10):

\[
\begin{align*}
\text{If } EI_{k-1} \leq s & \Rightarrow EI_k = EI_{k-1} + Q - D \\
\text{If } EI_{k-1} > s & \Rightarrow EI_k = EI_{k-1} - D
\end{align*} \quad (4.11)
\]

4.2.1. Mathematical Analysis

**Proposition:** \(EI\) can not indefinitely remain greater than order point \(s\). If \(EI > s \Rightarrow O = 0\) from Equation (4.10) and \(D > 0\). Therefore once effective inventory is above order point \(s\), it starts to decrease and it certainly drops to \(s\) or lower in a finite number of time steps; i.e. at some time point \((k-1)\), \(EI_{k-1} \leq s\) is certainly satisfied.

**Proposition:** \(EI\) indefinitely decreases if \(Q < D\). At some time point \((k-1)\), \(EI_{k-1} \leq s\) is certainly satisfied. From Equation (4.11), if \(EI_{k-1} \leq s \Rightarrow EI_k = EI_{k-1} + Q - D < EI_{k-1} \leq s\) since \(Q < D\), again from Equation (4.11) if \(EI_k < s \Rightarrow EI_{k+1} = EI_k + Q - D < EI_k < s\) since \(Q < D\) (To present a complete mathematical analysis of each policy under any condition, some conditions, which may be illogical to employ practically, are also analyzed; such as \(Q < D\)).

If \(EI\) indefinitely decreases, the order \(O_k\) stays constant at \(Q\) from Equation (4.10).

**Proposition:** \(EI\) stays constant at a value lower than or equal to order point \(s\) if \(Q = D\). At some time point \((k-1)\), \(EI_{k-1} \leq s\) is certainly satisfied. From Equation (4.11),
if \( EI_{k-1} \leq s \Rightarrow EI_k = EI_{k-1} + Q - D = EI_{k-1} \leq s \) since \( Q = D \), again from Equation (4.11) if \( EI_k \leq s \Rightarrow EI_{k+1} = EI_k + Q - D = EI_k = EI_{k-1} \leq s \) since \( Q = D \).

If effective inventory stays constant, the order \( O_k \) stays constant at \( Q \) from Equation (4.10).

**Proposition:** \( EI \) can not indefinitely remain on one side of order point \( s \) if \( Q > D \). At some time point \( (k-1) \), \( EI_{k-1} \leq s \) is certainly satisfied. From Equation (4.11), if \( EI_{k-1} \leq s \Rightarrow EI_k = EI_{k-1} + Q - D \). Since \( Q > D \), \( EI_k > EI_{k-1} \); effective inventory is increasing. Let \( l \) be the smallest integer greater than the ratio \( (s - EI_{k-1})/(Q - D) \). After \( l \) time points, \( EI_{k+l-1} = EI_{k-1} + l(Q - D) > s \) certainly from the definition of \( l \).

**Proposition:** For any \( Q = \frac{n}{l} D \) (i.e. \( \frac{Q}{D} = \frac{n}{l} \)) where \( \frac{n}{l} > 1 \) and \( n \) and \( l \) are relatively prime, \( EI \) displays period-\( n \) oscillation in which at \( l \) time points it is less than or equal to order point \( s \). This is proven in four stages.

Condition \( \frac{n}{l} > 1 \) guarantees \( Q > D \) so that effective inventory can not indefinitely remain on one side of order point \( s \). A cycle that repeats itself in \( n \), also repeats itself in \( 2n, 3n, \ldots \) and period is the smallest repetition cycle. The condition \( n \) and \( l \) are relatively prime, which is elaborated in Stage Four, guarantees repetition length \( n \) of effective inventory is the smallest, i.e. the period.

While writing behavior sequence of effective inventory at consecutive time points starting from an arbitrary time point \( (k-1) \), from Equation (4.11), a \( Q \) term is introduced when effective inventory drops to \( s \) or lower, from both Equations (4.11) and (4.12), a \(-D\) term is introduced at every time point other than the first one (since at the first time point, effective inventory is an independent initial condition).

**First Stage:** For any value of \( n \) and \( l \), effective inventory drops to \( s \) or lower at least \( l \) times in at most \( n \) time points.
The condition for effective inventory stated in Equation (4.13) is certainly satisfied for some \((k-1)\), since if \(Q > D\) it is proved that effective inventory can not indefinitely remain on one side of order point \(s\). When effective inventory is greater than order point \(s\), \((s, s + Q - D]\) is the maximum interval it can take values in.

\[
s < EI_{k-1} \leq s + Q - D
\]  \hspace{1cm} (4.13)

*EI drops to \(s\) or lower at least one time at most in \((n-l+1)\) time points.*

\[
s < EI_{k-1} \leq s + Q - D
\]

\[
s - (n-l)D < EI_{k-1} - (n-l)D \leq s + Q - D - (n-l)D
\]

\[
s - (n-l)D < EI_{k+n-l-1} \leq s + \frac{n}{l}D - (n-l+1)D
\]

\[
s - (n-l)D < EI_{k+n-l-1} \leq s - \frac{(n-l)(l-1)}{l}D
\]

\[
(n-l)(l-1)/l \geq 0 \text{ for } n \geq l \text{ and } l \geq 1
\]

*EI drops to \(s\) or lower at least two times at most in \((n-l+2)\) time points:* From above, although not known at which time, *EI drops to \(s\) or lower at least one time at most in \((n-l+1)\) time points.*

\[
s < EI_{k-1} \leq s + Q - D
\]

\[
s + Q - (n-l+1)D < EI_{k-1} + Q - (n-l+1)D \leq s + Q - D + Q - (n-l+1)D
\]

\[
s + \frac{n}{l}D - (n-l+1)D < EI_{k+n-l} \leq s + \frac{2n}{l}D - (n-l+2)D
\]

\[
s - \frac{(n-l)(l-1)}{l}D < EI_{k+n-l} \leq s - \frac{(n-l)(l-2)}{l}D
\]

\[
(n-l)(l-2)/l \geq 0 \text{ for } n \geq l \text{ and } l \geq 2
\]
EI drops to \( s \) or lower at least \((l - 1)\) times at most in \((n - l + (l - 1)) = n - 1\) time points: From above, although not known at which time, \( EI \) drops to \( s \) or lower at least \((l - 2)\) times at most in \((n - l + (l - 2)) = n - 2\) time points.

\[
s < EI_{k-1} \leq s + Q - D \\
s + (l - 2)Q - (n - 2)D < EI_{k-1} + (l - 2)Q - (n - 2)D \leq s + Q - D + (l - 2)Q - (n - 2)D \\
s + (l - 2)\frac{n}{l}D - (n - 2)D < EI_{k+n-3} \leq s + (l - 1)\frac{n}{l}D - (n - 1)D \\
s - \frac{2(n - l)}{l}D < EI_{k+n-3} \leq s - \frac{n - l}{l}D \\
\frac{(n - l)}{l} \geq 0 \text{ for } n \geq l
\]

\( EI \) drops to \( s \) or lower at least \( l \) times in at most \( n \) time points: From above, although not known at which time, \( EI \) drops to \( s \) or lower at least \((l - 1)\) times at most in \((n - 1)\) time points.

\[
s < EI_{k-1} \leq s + Q - D \\
s + (l - 1)Q - (n - 1)D < EI_{k-1} + (l - 1)Q - (n - 1)D \leq s + Q - D + (l - 1)Q - (n - 1)D \\
s + (l - 1)\frac{n}{l}D - (n - 1)D < EI_{k+n-2} \leq s + l\frac{n}{l}D - nD \\
s - \frac{n - l}{l}D < EI_{k+n-2} \leq s
\]

As a result, effective inventory drops to \( s \) or lower at least \( l \) times at most in \( n \) time points.

Second Stage: After \( EI \) drops to \( s \) or lower \( l \) times, it stays greater than \( s \) until the end of \( n \) time points.

Assume at any \( j^{th} \) time point from \((k - 1)\), where \( l \leq j \leq n \), \( EI \) drops to \( s \) or lower \( l \) times ((k - 1) is the first time point, \( k \) is the second and so on, therefore \( j^{th} \) time point means \( k - 1 + (j - 1) = k + j - 2 \).
\[ EI_{k+j-2} = EI_{k-1} + lQ - (j-1)D = EI_{k-1} + \frac{n}{l}D - (j-1)D = EI_{k-1} + nD - (j-1)D \]

\[ EI_{k+j-2} = EI_{k-1} + (n - j + 1)D \]

Since \( l \leq j \leq n \Rightarrow (j-1) < n \Rightarrow (n - j + 1) > 0 \). Add \((n - j + 1)D\) to Equation (4.13).

\[ s < EI_{k-1} \leq s + Q - D \]

\[ s + (n - j + 1)D < EI_{k-1} + (n - j + 1)D \leq s + Q - D + (n - j + 1)D \]

\[ s + (n - j + 1)D < EI_{k+j-2} \leq s + Q + (n - j)D \]

\[ EI_{k+j-2} > s \quad \text{for} \quad l \leq j \leq n \quad (4.14) \]

As a result, from Equation (4.14), it is seen that \( EI \) stays greater than order point \( s \) up to \( n^{th} \) time point from \((k-1)\), after it drops to \( s \) or lower \( l \) times up to \( j^{th} \) time point from \((k-1)\), where \( l \leq j \leq n \). Considering results of first and second stages together, \( EI \) drops to \( s \) or lower \( l \) times in \( n \) time points.

**Third Stage:** Effective inventory repeats itself in each \( n \), i.e. \( EI_{k+n-1} = EI_{k-1} \). From first and second stages, \( EI \) drops to order point \( s \) or lower \( l \) times in \( n \) time points.

\[ EI_{k+n-1} = EI_{k-1} + lQ - nD = EI_{k-1} + \frac{n}{l}D - nD \]

\[ EI_{k+n-1} = EI_{k-1} \]

**Fourth Stage:** \( EI \) does not just repeat in \( n \) but also displays period-\( n \) oscillation (The repetition length \( n \) is the smallest, i.e. the period).

Define \( a \) as the period and \( b \) as the number of times effective inventory drops to \( s \) or lower in a period. At \((a+1)^{th}\) time point starting from \((k-1)\), effective inventory satisfies the following equation.
\[ EI_{k+a-1} = EI_{k-1} + bQ - aD \]

From definition of periodicity, \( EI_{k+a-1} = EI_{k-1} \). We search for \( a < n \) and \( b < l \) which satisfy Equation (4.15).

\[ bQ - aD = 0 \Rightarrow b \frac{n}{l} D - aD = 0 \]

\[ b \frac{n}{l} - a = 0 \]  \hspace{1cm} (4.15)

From definitions of \( a \) and \( b \), they have to be integers. Since \( n \) and \( l \) are relatively prime, the first \( a \) and \( b \) values that satisfy Equation (4.15) are \( a = n \) and \( b = l \). Therefore, \( n \) is not only any repetition length but also the period.

If effective inventory displays period-\( n \) oscillation so does the order \( O_k \) from Equation (4.10).

If the relationship between \( Q \) and \( D \) is not representable by rational numbers, for example \( Q = \sqrt{2}D \), though rigorous analysis is not made, the following logic seems reasonable and it is supported by simulation results. Any irrational number can be approximated to a certain extent by a rational number. After approximating the irrational number by a rational number whose numerator and denominator are relatively prime, effective inventory displays period-numerator oscillation (the better the approximation, the better the results).

If effective inventory displays period-numerator oscillation so does the order \( O_k \) from Equation (4.10).

To summarize, effective inventory indefinitely decreases and order exhibits goal seeking behavior if \( Q < D \); they both exhibit goal seeking behavior if \( Q = D \). If \( Q/D > 1 \) and rational, after simplifying \( Q/D \) until numerator and denominator are relatively prime, they both display period-numerator oscillation and lastly if \( Q/D > 1 \) and irrational, after
approximating \( Q/D \) by a rational number whose numerator and denominator are relatively prime, they both display period-numerator oscillation.

4.2.2. Simulation Experiments

Example 1: Consider parameters in Table 4.4. They satisfy \( Q = 1 < D = 2 \) therefore effective inventory must indefinitely decrease and orders must exhibit goal seeking behavior as confirmed in Figure 4.4.

Table 4.4. \((s, Q)\) parameters yielding decreasing \( EI\) and constant order behavior

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>( D )</th>
<th>( Q )</th>
<th>( \tau )</th>
<th>( s )</th>
<th>( I_0 )</th>
<th>( SL_0 )</th>
<th>( s+Q-D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

![Graph](image)

Figure 4.4. \((s, Q)\) policy, decreasing \( EI\) and constant order behavior

Example 2: Consider parameters in Table 4.5. They satisfy \( Q = D = 4 \) therefore goal seeking behavior must result as confirmed in Figure 4.5.

Table 4.5. \((s, Q)\) parameters yielding goal seeking behavior

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>( D )</th>
<th>( Q )</th>
<th>( \tau_1 )</th>
<th>( \tau_2 )</th>
<th>( s )</th>
<th>( I_0 )</th>
<th>( SL_0 )</th>
<th>( SL_0' )</th>
<th>( s+Q-D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>32</td>
<td>15</td>
<td>16</td>
<td>20</td>
<td>32</td>
<td></td>
</tr>
</tbody>
</table>
Figure 4.5. \((s,Q)\) policy, second order continuous delay, goal seeking behavior

Example 3: Consider parameters in Table 4.6. They satisfy \(Q/D = 8/3\) therefore period-8 oscillation, at 3 time points effective inventory is at a lower than order point \(s = 39\) in a period, must result as confirmed in Figure 4.6.

Table 4.6. \((s,Q)\) parameters yielding P-8 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>(Q = (8/3)D)</th>
<th>(T_1)</th>
<th>(T_2)</th>
<th>(T_3)</th>
<th>(s)</th>
<th>(I_0)</th>
<th>(SL_1)</th>
<th>(SL_2)</th>
<th>(SL_3)</th>
<th>(s+Q-D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>3</td>
<td>3</td>
<td>8</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>39</td>
<td>40</td>
<td>0</td>
<td>0</td>
<td>44</td>
</tr>
</tbody>
</table>

Figure 4.6. \((s,Q)\) policy, third order continuous delay, P-8 oscillation

Example 4: Consider parameters in Table 4.7. They satisfy \(Q/D = 5/2\) therefore period-5 oscillation, at 2 time points \(EI\) is at a lower than order point \(s = 12.75\) in a
period, must result as confirmed in Figure 4.7. In this example, first and second delay structures are continuous, while third one is discrete.

Table 4.7. \((s, Q)\) parameters with mixed delay yielding P-5 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>(D)</th>
<th>(Q=(5/2)D)</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>(\tau_3)</th>
<th>(s)</th>
<th>(I_0)</th>
<th>(SL_0^1)</th>
<th>(SL_0^2)</th>
<th>(SL_0^3)</th>
<th>(s+Q-D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>3</td>
<td>1.5</td>
<td>3.75</td>
<td>2.5</td>
<td>3</td>
<td>2</td>
<td>12.75</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>15</td>
</tr>
</tbody>
</table>

![Figure 4.7. \((s, Q)\) policy, third order mixed delay, P-5 oscillation](image)

Example 5: Consider parameters in Table 4.8. They satisfy \(Q/D = \sqrt{14} \approx 15/4\) where \%error=\%0.223, therefore period-15 oscillation, at 4 time points \(EI\) is at or lower than order point \(s = 4\) in a period, must result as confirmed in Figure 4.8.

Table 4.8. \((s, Q)\) parameters yielding P-15 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>(D)</th>
<th>(Q=\sqrt{14}D)</th>
<th>(s)</th>
<th>(I_0)</th>
<th>(SL_0)</th>
<th>(s+Q-D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>1</td>
<td>1</td>
<td>(\sqrt{14})</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>
Figure 4.8. \((s,Q)\) policy, first order delay, irrational ratio, \(P-15\) oscillation

4.3. Effective Inventory and Orders Dynamics under \((R,S)\) Policy

Review Period - Order Up to Level rule is mathematically expressed as:

\[
O_k = S - EI_k \quad \text{if } (\text{mod}(k,R) = 0)
\]

\[
0 \quad \text{else}
\]

\[
(4.16)
\]

From Equations (4.2) and (4.16):

\[
\text{If } \text{mod}(k,R) = 0 \Rightarrow EI_{k+1} = S - D
\]

\[
(4.17)
\]

\[
\text{If } \text{mod}(k,R) \neq 0 \Rightarrow EI_{k+1} = EI_k - D
\]

\[
(4.18)
\]

4.3.1. Mathematical Analysis

Proposition: Effective inventory exhibits goal seeking behavior and stays constant at \(S - D\) if \(R = 1\). The if condition in Equation (4.16) is always satisfied for \(R = 1\), thus from Equation (4.17) effective inventory is always \(S - D\).
If effective inventory stays constant at $S - D$, the order $O_k$ stays constant at $D$ from Equation (4.16).

**Proposition:** For any integer $R > 1$, effective inventory displays period-$R$ oscillation. If we write behavior of effective inventory using Equations (4.17) and (4.18) at consecutive time points, period-$R$ oscillation is easily seen.

$$\mod(Rk, R) = 0 \Rightarrow EI_{rk+1} = S - D$$
$$\mod(Rk + 1, R) \neq 0 \Rightarrow EI_{rk+2} = EI_{rk+1} - D = S - 2D$$
$$\mod(Rk + 2, R) \neq 0 \Rightarrow EI_{rk+3} = EI_{rk+2} - D = S - 3D$$
$$\vdots$$
$$\mod(Rk + R - 1, R) \neq 0 \Rightarrow EI_{rk+R} = EI_{rk+R-1} - D = S - RD$$
$$\mod(Rk + R, R) = 0 \Rightarrow EI_{rk+R+1} = S - D$$

Whenever time is a multiple of $R$, from Equation (4.17), the next time point it goes to $S - D$. From Equation (4.18), effective inventory decreases until time is again a multiple of $R$ and length between two points which are multiples of $R$ is simply $R$. If you start from the same constant $S - D$ and decrease by a positive constant $D$ during $R$ time points you certainly pass from the same points.

If effective inventory displays period-$R$ oscillation so does the order $O_k$ from Equation (4.16). Since time is a multiple of review period $R$ only once in a period, order decision is made once (at this point the order quantity is $S - EI = S - (S - RD) = RD$) in a period and at other points in that period the order $O_k$ is zero. Thus, order displays period-$R$ oscillation with points $(RD, 0, \ldots, 0)$ where number of zeros is $R - 1$.

Up to now, $R$ is assumed to be positive integer values. What happens if $R$ is not an integer number? First of all, $R$ can not be zero or a negative number. Secondly, $R$ must be greater than or equal to 1, since it is illogical to consider a review period which is smaller than simulation time step. Now assuming $R \geq 1$, there are two alternatives; either $R$ is a rational number or $R$ is an irrational number. Since irrational numbers can not be written exactly as rational numbers and since the if condition in Equation (4.16) is exact, it can not
be satisfied at any time point. Thus $EI$ indefinitely drops and order amount is always zero. There remains just rational $R$ case to be analyzed.

*Proposition*: For any $R = a/b$, where $a/b > 1$ and $a$ and $b$ are relatively prime $EI$ exhibits exactly the same behavior as $R = a$ (In both cases effective inventory displays period-$a$ oscillation).

$R$ is known to be a number greater than 1. Express it as a ratio whose numerator and denominator are relatively prime. We call it $R = a/b$ ($a/b > 1$ since $R > 1$). Now, we search for time points $k$ which are multiples of $R = a/b$.

$$k/R = k/(a/b) = (b \cdot k)/a$$ (4.19)

Thus, time points multiples of $R = a/b$ are $k = 0, a, 2a, \ldots$ from Equation (4.19). Time points multiples of $R = a$ are also $k = 0, a, 2a, \ldots$ Therefore, a time point which is a multiple of $a$ is also a multiple of $a/b$. As a result of this, they show exactly the same behavior and since it is known that when $R = a$ effective inventory displays period-$a$ oscillation, so does $R = a/b$ (In fact the discussion is more general, exact behavior is not restricted to the dynamics of effective inventory. $R = a/b$ and $R = a$ have exactly the same system equations so not only effective inventories but all dynamics are same).

If discrete time step, $k$, is specified as a small enough step, to choose $R = a/b$ does not have any meaning. This case is analyzed just for mathematical completeness. That is why $R = a/b$ and $R = a$ produce exactly the same dynamics which is again not a meaningful result.

To summarize, effective inventory and order exhibit goal seeking behavior if $R = 1$. If $R > 1$ and integer, they both display period-$R$ oscillation.
4.3.2. Simulation Experiments

Example 1: Consider parameters in Table 4.9. They satisfy $R = 1$ therefore goal seeking behavior must result as confirmed in Figure 4.9.

Table 4.9. $(R, S)$ parameters yielding goal seeking behavior

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>$R$</th>
<th>$D$</th>
<th>$\tau$</th>
<th>$S$</th>
<th>$I_0$</th>
<th>$SL_0$</th>
<th>S-D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>2</td>
<td>20</td>
<td>0</td>
<td>0</td>
<td>15</td>
</tr>
</tbody>
</table>

Figure 4.9. $(R, S)$ policy, first order continuous delay, goal seeking behavior

Example 2: Consider parameters in Table 4.10. They satisfy $R = 3/2$ therefore period-3 oscillation must result as confirmed in Figure 4.10. As stated above, to choose $R = 3/2$ is not meaningful, if discrete time step, $k$, is specified as a small enough step.

Table 4.10. $(R, S)$ parameters yielding P-3 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>$R$</th>
<th>$D$</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$S$</th>
<th>$I_0$</th>
<th>$SL_0$</th>
<th>$SL_0'$</th>
<th>S-D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>2</td>
<td>3/2</td>
<td>4</td>
<td>5</td>
<td>2</td>
<td>38</td>
<td>10</td>
<td>13</td>
<td>16</td>
<td>34</td>
</tr>
</tbody>
</table>
Figure 4.10. \((R, S)\) policy, second order continuous delay, P-3 oscillation

Example 3: Consider parameters in Table 4.11. They satisfy \(R = 4\) therefore period-4 oscillation must result as confirmed in Figure 4.11. In this example, first and third delays are discrete, while second one is continuous.

Table 4.11. \((R, S)\) parameters with mixed delay yielding P-4 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>(R)</th>
<th>(D)</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>(\tau_3)</th>
<th>(S)</th>
<th>(I_0)</th>
<th>(S/L_0)</th>
<th>(S/L_1)</th>
<th>(S/L_2)</th>
<th>(S-D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>11</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

Figure 4.11. \((R, S)\) policy, third order mixed delay, P-4 oscillation
4.4. Effective Inventory and Orders Dynamics under \((R,s,S)\) Policy

\((R,s,S)\) rule is mathematically expressed as:

\[
O_k = \begin{cases} 
S - EI_k & \text{if } \left( \mod(k,R) = 0 \text{ and } EI_k \leq s \right) \\
0 & \text{else}
\end{cases}
\]  \hspace{1cm} (4.20)

From Equations (4.2) and (4.20):

\[
\text{If } \left( \mod(k,R) = 0 \text{ and } EI_k \leq s \right) \Rightarrow EI_{k+1} = S - D
\]  \hspace{1cm} (4.21)

\[
\text{If } \left( \mod(k,R) \neq 0 \text{ and/or } EI_k > s \right) \Rightarrow EI_{k+1} = EI_k - D
\]  \hspace{1cm} (4.22)

4.4.1. Mathematical Analysis

\((R,s,S)\) policy is a combination of \((s,S)\) and \((R,S)\). Since enough experience is built on \((s,S)\) and \((R,S)\) policies, we present a rather intuitive discussion using what we learnt while studying \((R,S)\) and \((s,S)\) and give detailed mathematical analysis in Appendix A.

If \(R=1\), first condition in if statement in Equation (4.20) is always satisfied and \((R,s,S)\) completely turns into \((s,S)\) and all the results in \((s,S)\) are valid.

If \((S-s) \leq D\) (or \((S-D) \leq s\)), effective inventory is always less than or equal to \(s\) since \(S-D\) is the maximum value effective inventory can take. Therefore second condition in the if statement in Equation (4.20) is always satisfied. As a result, \((R,s,S)\) completely turns into \((R,S)\), and all the results in \((R,S)\) are valid.

From definition of \(S\) in Section 3.4, \((S-s) \leq D\) can not be satisfied. Formula of order up to level \(S\) places a restriction in the analysis. In order to carry out a more general analysis, \(S\) is calculated by adding a positive constant to order point \(s\) in all chapters and
simulation experiments. This change just affects the average level of dynamics and it does not make a qualitative difference in the dynamics.

\[ R > 1 \text{ and } (S - s) > D \] is a new case. Effective inventory under \((R, S)\) policy displays period-\(R\) oscillation. However in \((R, s, S)\) policy, effective inventory is also checked against order point \(s\) at time points multiples of \(R\) so it is intuitive that effective inventory under \((R, s, S)\) displays oscillatory behavior in a multiple of \(R\).

To summarize (results from Appendix A), effective inventory and order exhibit goal seeking behavior if \(R = 1\) and \((S - s) \leq D\). If integer \(R > 1\) and/or \((S - s) > D\), they both display period-\(nR\) oscillation where \(n\) is the smallest integer greater than or equal to the ratio \((S - s)/(RD)\).

### 4.4.2. Simulation Experiments

Example 1: Consider parameters in Table 4.12. They satisfy \(R = 1\) and \((S - s) = 3 \leq D = 5\) therefore goal seeking behavior must result as confirmed in Figure 4.12.

**Table 4.12. \((R, s, S)\) policy yielding goal seeking behavior**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>R</th>
<th>D</th>
<th>(t)</th>
<th>s</th>
<th>S</th>
<th>(l_0)</th>
<th>SL_0</th>
<th>S-D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td></td>
<td>1</td>
<td>5</td>
<td>2</td>
<td>20</td>
<td>23</td>
<td>10</td>
<td>0</td>
<td>18</td>
</tr>
</tbody>
</table>

![Figure 4.12. \((R, s, S)\) policy, first order continuous delay, goal seeking behavior](image-url)
Example 2: Consider parameters in Table 4.13. They satisfy $R = 1 \Rightarrow$ exactly $(S - s) \Rightarrow (S - s)/D = 11/4 \leq n \Rightarrow n = 3$ therefore period-3 oscillation must result as confirmed in Figure 4.13.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>$R$</th>
<th>$D$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$s$</th>
<th>$S$</th>
<th>$I_0$</th>
<th>$SL_0$</th>
<th>$SL_0'$</th>
<th>$S-D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>28</td>
<td>39</td>
<td>10</td>
<td>10</td>
<td>19</td>
<td>35</td>
</tr>
</tbody>
</table>

Figure 4.13. $(R, s, S)$ policy, second order continuous delay, P-3 oscillation

Example 3: Consider parameters in Table 4.14. They satisfy $R = 2$ and $(S - s) = 5 > D = 2 \Rightarrow (S - s)/RD = 5/4 \leq n \Rightarrow n = 2$ therefore period-4 oscillation must result as confirmed in Figure 4.14. In this example, first and second delays are discrete, while third one is continuous.

Table 4.14. $(R, s, S)$ parameters with mixed delay yielding P-4 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Order of delay</th>
<th>$R$</th>
<th>$D$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_3$</th>
<th>$s$</th>
<th>$S$</th>
<th>$I_0$</th>
<th>$SL_0$</th>
<th>$SL_0'$</th>
<th>$SL_0''$</th>
<th>$S-D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>30</td>
<td>35</td>
<td>25</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>33</td>
</tr>
</tbody>
</table>
Figure 4.14. \((R,s,S)\) policy, third order mixed delay, P-4 oscillation

Note that effective inventory is at order point \(s\) or lower more than one time in a period but only one of them is a multiple of \(R = 2\).

4.5. Chapter Conclusion

The following results are valid for any order of supply line and delay type (even for any flow equation, where demand is a positive constant).

Under \((s,S)\) policy, effective inventory and order exhibit goal seeking behavior if \((S-s) \leq D\) and they display period-\(n\) oscillation if \((S-s) > D\) where \(n\) is the smallest integer greater than or equal to ratio \((S-s)/D\).

Under \((s,Q)\) policy, effective inventory indefinitely decreases and order exhibits goal seeking behavior if \(Q < D\); they both exhibit goal seeking behavior if \(Q = D\). If \(Q/D > 1\) and rational, after simplifying \(Q/D\) until numerator and denominator are relatively prime, they both display period-numerator oscillation. Lastly, if \(Q/D > 1\) and irrational, after approximating \(Q/D\) by a rational number whose numerator and denominator are relatively prime, they both display period-numerator oscillation.
Under \((R, S)\) policy, effective inventory and order exhibit goal seeking behavior if \(R = 1\). If integer \(R > 1\), they both display period-\(R\) oscillation.

Under \((R, s, S)\) policy, effective inventory and order exhibit goal seeking behavior if \(R = 1\) and \((S - s) \leq D\). If integer \(R > 1\) and/or \((S - s) > D\), they both display period-\(nR\) oscillation where \(n\) is the smallest integer greater than or equal to the ratio \((S - s)/(RD)\).

Table 4.15 summarizes how effective inventory and orders behave under different conditions for all inventory policies.

<table>
<thead>
<tr>
<th>Policy</th>
<th>Condition</th>
<th>Behavior of effective inventory and orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>((s, S))</td>
<td>((S - s) \leq D)</td>
<td>Goal seeking</td>
</tr>
<tr>
<td></td>
<td>((S - s) &gt; D)</td>
<td>Period-(n) oscillation where (n) is the smallest integer greater than or equal to the ratio ((S - s)/D)</td>
</tr>
<tr>
<td>((s, Q))</td>
<td>(Q &lt; D)</td>
<td>Effective inventory ever decreasing; order constant</td>
</tr>
<tr>
<td></td>
<td>(Q = D)</td>
<td>Goal seeking</td>
</tr>
<tr>
<td></td>
<td>Rational (Q/D &gt; 1)</td>
<td>Period-numerator oscillation</td>
</tr>
<tr>
<td></td>
<td>(simplify (Q/D) until numerator and denominator are relatively prime)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Irrational (Q/D &gt; 1)</td>
<td>Approximately period-numerator oscillation</td>
</tr>
<tr>
<td></td>
<td>(approximate by a rational number whose numerator and denominator are relatively prime)</td>
<td></td>
</tr>
<tr>
<td>((R, S))</td>
<td>(R = 1)</td>
<td>Goal seeking</td>
</tr>
<tr>
<td></td>
<td>Integer (R &gt; 1)</td>
<td>Period-(R) oscillation</td>
</tr>
<tr>
<td>((R, s, S))</td>
<td>(R = 1) and ((S - s) \leq D)</td>
<td>Goal seeking</td>
</tr>
<tr>
<td></td>
<td>Integer (R &gt; 1) and/or ((S - s) &gt; D)</td>
<td>Period-(nR) oscillation where (n) is the smallest integer greater than or equal to the ratio ((S - s)/(RD))</td>
</tr>
</tbody>
</table>

In all inventory policies, order shows just two behaviors in different conditions; either it exhibits goal seeking behavior or periodic behavior. In all situations except one,
effective inventory and order exhibit the same behavior. Only in \((s,Q)\) policy when \(Q < D\), effective inventory indefinitely decreases while order exhibits goal seeking behavior.

Orders behavior in \((s,S)\), \((R,S)\) and \((R,s,S)\) policies is very similar. In these three policies, order decision is made just once in any period with quantity \(PD\) and at other points of that period the order \(O_k\) is zero; order displays period-\(P\) oscillation with points \((PD,0,...,0)\) where number of zeros is \(P-1\) (\(P\) is period, in \((s,S)\) \(P = n\), in \((R,S)\) \(P = R\) and in \((R,s,S)\) \(P = nR\)).
5. INVENTORY DYNAMICS FORMULAS WITH FIRST ORDER DELAYS

In this chapter, dynamics of inventory with first order continuous delay \((M = 1)\) under different policies are analyzed. For all inventory policies receiving delay \(\tau\) is assumed to be greater than 1 \((\tau \geq 1)\), otherwise outflow of supply line stock exceeds its level (since the model is discrete) which is illogical.

From Chapter 3, it is known that Stock Acquisition sector for all inventory policies is the same so independent of the inventory policy, it is represented by the following set of equations.

\[
\begin{align*}
I_k &= I_{k-1} + SL_{k-1}/\tau - D \\
SL_k &= SL_{k-1} + O_{k-1} - SL_{k-1}/\tau
\end{align*}
\]  
\(5.1\)

5.1. Order Point - Order Up to Level \((s, S)\) Policy

Order Point-Order Up to Level rule is mathematically expressed as:

\[
O_k = S - EI_k \quad \text{if} \ (EI_k \leq s) \\
0 \quad \text{else}
\]  
\(5.2\)

5.1.1. Goal Seeking Behavior

Assume system parameters satisfy \((S - s) \leq D\). From Section 4.1.1, if \((S - s) \leq D \Rightarrow EI_k \leq s\) and from Equation (5.2), \(O_k = S - EI_k\) in the long run, therefore using Equation (5.1) system equations are as follows:

\[
\begin{align*}
I_k &= I_{k-1} + SL_{k-1}/\tau - D \\
SL_k &= SL_{k-1} + S - I_{k-1} - SL_{k-1}/\tau
\end{align*}
\]  
\(5.3\)
Equilibrium points of the system defined by Equation (5.3) are:

\[ I_k - I_{k-1} = SL_{k-1} / \tau - D = 0 \Rightarrow SL_e = \tau D \]

\[ SL_e - SL_{k-1} = S - I_{k-1} - SL_{k-1} - SL_{k-1} / \tau = 0 \Rightarrow S - I_e - SL_e - SL_e / \tau = 0 \Rightarrow I_e = S - (\tau + 1)D \]

Thus, \( SL_e = \tau D \) and \( I_e = S - (\tau + 1)D \) are equilibrium points of supply line and inventory respectively.

In order to carry out stability analysis, disregard constant terms in Equation (5.3) without loss of generality since they do not affect stability.

\[ B = \begin{pmatrix} 1 & 1/\tau \\ -1 & -1/\tau \end{pmatrix} \Rightarrow |B - \lambda I| = \begin{vmatrix} 1 - \lambda & -1 \\ 1/\tau & -1/\tau - \lambda \end{vmatrix} = \lambda(\lambda + 1/\tau - 1) = 0 \Rightarrow \begin{cases} \text{either } \lambda = 0 \\ \text{or } \lambda = 1 - 1/\tau \end{cases} \]

Since roots of characteristic equation are in the unit circle (remember that \( \tau \geq 1 \)), if \((S - s) \leq D\) both supply line and inventory have stable equilibrium points.

To summarize, if \((S - s) \leq D\) is satisfied, supply line and inventory exhibit goal seeking behavior with equilibrium points \( SL_e = \tau D \) and \( I_e = S - (\tau + 1)D \) respectively.

Simulation experiment: Consider parameters in Table 5.1. They satisfy \((S - s) = 10 \leq D = 20\) therefore goal seeking behavior must result as confirmed in Figure 5.1 and Table 5.2.

| Table 5.1. \((s, S)\) parameters yielding goal seeking behavior |
|-----------------|-----|-----|-----|-----|-----|
| Parameters      | D   | Q   | \(t\) | \(s\) | \(S\) | \(I_0\) | \(SL_0\) |
| Values          | 20  | 10  | 4    | 100 | 110 | 0     | 0      |
Table 5.2. \((s, S)\) policy, goal seeking behavior, analytical and simulation results

<table>
<thead>
<tr>
<th>Time</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical results</td>
<td>Supply line</td>
</tr>
<tr>
<td></td>
<td>Inventory</td>
</tr>
<tr>
<td>Simulation results</td>
<td>Supply line</td>
</tr>
<tr>
<td></td>
<td>Inventory</td>
</tr>
</tbody>
</table>

Figure 5.1. \((s, S)\) policy, first order continuous delay, goal seeking behavior

Note that in Figure 5.1, since points are connected, especially goal seeking behavior graphs may give the wrong impression of continuous time simulations. All simulation experiments in this thesis, except those in Appendix C, are discrete time simulations, time step being one.

5.1.2. Periodic Behaviors

From Section 4.1.1, it is known that if \((S - s) > D\) then \(EI\) displays periodic oscillation in which at only one time point it drops to order point \(s\) or lower in a period. Using this, the following 3-step procedure is proposed to analyze the system: First, a behavior sequence of effective inventory (inventory position) is stated, secondly corresponding system equations are written for this behavior sequence and lastly these
system equations are analyzed under the stated behavior sequence to find under what condition they are noncontradictory.

Sequence Possibilities: For a period of length \( n \) where effective inventory drops to order point \( s \) or lower once in a period there may be \( C(n, 1) = n \) different behavior sequences where \( C \) represents combination symbol. According to the initial point, there can be \( n \) different views of a period of length \( n \). Since possible different views of the behavior sequence is equal to possible number of behavior sequences, all different sequences are equivalent (they produce exactly the same dynamics).

Simulation Experiment: Consider parameters in Table 5.3. In Table 5.4 effective inventory dynamics sequence during three periods are shown, from time 58 to 66.

Table 5.3. \((s, S)\) parameters for sequence illustration

<table>
<thead>
<tr>
<th>Parameters</th>
<th>D</th>
<th>Q</th>
<th>T</th>
<th>s</th>
<th>S</th>
<th>(I_0)</th>
<th>(SL_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>10</td>
<td>25</td>
<td>4</td>
<td>50</td>
<td>75</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5.4. \((s, S)\) policy, \(EI\) dynamics sequence illustration

<table>
<thead>
<tr>
<th>Time</th>
<th>58</th>
<th>59</th>
<th>60</th>
<th>61</th>
<th>62</th>
<th>63</th>
<th>64</th>
<th>65</th>
<th>66</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effective Inventory</td>
<td>(65&gt;s)</td>
<td>(55&gt;s)</td>
<td>(45 \leq s)</td>
<td>(65&gt;s)</td>
<td>(55&gt;s)</td>
<td>(45 \leq s)</td>
<td>(65&gt;s)</td>
<td>(55&gt;s)</td>
<td>(45 \leq s)</td>
</tr>
</tbody>
</table>

We observe the cycle of length 3 in Table 5.4 starting from time 58, \(EI_{58} > s, EI_{59} > s, EI_{60} \leq s\), this repeats itself (Period-3). We observe the cycle of length 3 in Table 5.4 starting from time 59, \(EI_{59} > s, EI_{60} \leq s, EI_{61} > s\), this repeats itself (Period-3). We observe the cycle of length 3 in Table 5.4 starting from time 60, \(EI_{60} \leq s, EI_{61} > s, EI_{62} > s\), this repeats itself (Period-3).

As a result, three possible sequences are obtained from the same simulation. Therefore, all three are equivalent. Equivalency means taking any of those proposed sequences produces the same conclusions about the system.
For the remaining analysis of \((s,S)\) policy periodic behaviors, the following sequence convention is adopted: At the last point of the period, effective inventory drops to order point \(s\) or lower.

**Period-2 Oscillation**

If state of \(EI\) is assumed as Equation (5.4) at time \((k-1)\), using Equations (5.1) and (5.2) state of the system can be characterized by the following equations.

\[
EI_{k-1} > s \quad (5.4)
\]
\[
I_k = I_{k-1} + SL_{k-1}/\tau - D \quad (5.5)
\]
\[
SL_k = SL_{k-1} - SL_{k-1}/\tau \quad (5.6)
\]

If state of \(EI\) is assumed as Equation (5.7) at time \(k\), using Equations (5.1) and (5.2) state of the system can be characterized by the following equations.

\[
EI_k \leq s \quad (5.7)
\]
\[
I_{k+1} = I_k + SL_k/\tau - D \quad (5.8)
\]
\[
SL_{k+1} = SL_k + SL_k - SL_k/\tau \quad (5.9)
\]

Under what condition does the system display period-2 oscillation \((I_{k+1} = I_{k-1}\) and \(SL_{k+1} = SL_{k-1}\)\)?

From Equations (5.5) and (5.8):

\[
I_{k+1} = I_{k-1} + \frac{SL_{k-1}}{\tau} - D + \frac{SL_k}{\tau} - D = I_{k-1} + \frac{SL_{k-1} + SL_k}{\tau} - 2D \Rightarrow I_{k-1} = I_{k-1} + \frac{SL_{k-1} + SL_k}{\tau} - 2D
\]
\[
SL_{k+1} + SL_k = 2\tau D \quad (5.10)
\]

From Equations (5.6), (5.9) and (5.10):
\[ SL_{k+1} = SL_{k-1} - \frac{SL_{k-1}}{\tau} + S - I_k - SL_k - \frac{SL_k}{\tau} = SL_{k-1} - \frac{SL_{k-1} + SL_k}{\tau} + S - I_k - SL_k \]

\[ SL_{k-1} = SL_{k-1} - \frac{SL_{k-1} + SL_k}{\tau} + S - I_k - SL_k \Rightarrow 2D = S - I_k - SL_k \]

\[ EI_k = S - 2D \quad (5.11) \]

If Equations (5.5) and (5.6) are added, \( EI_k = EI_{k-1} - D \). Using \( EI_k = S - 2D \):

\[ S - 2D = EI_{k-1} - D \Rightarrow EI_{k-1} = S - D \quad (5.12) \]

Finally two equations, Equations (5.11) and (5.12), that are in the form of behavior of effective inventory, Equations (5.7) and (5.4), are obtained. Considering these four equations we find the condition for the system to display period-2 oscillation.

From Equations (5.4) and (5.12):

\[ \begin{align*}
EI_{k-1} &> s \\
EI_{k-1} &= S - D
\end{align*} \]

\( \Rightarrow (S - D) > s \Rightarrow D < (S - s) \quad (5.13) \)

From Equations (5.7) and (5.11):

\[ \begin{align*}
EI_k &\leq s \\
EI_k &= S - 2D
\end{align*} \]

\( \Rightarrow (S - 2D) \leq s \Rightarrow (S - s) \leq 2D \quad (5.14) \)

From Equations (5.13) and (5.14):

\[ D < S - s \leq 2D \quad (5.15) \]

If the relationship between \( s \) and \( S \) satisfies Equation (5.15) then the system (both supply line and inventory stocks) displays period-2 oscillation.
Supply line values can be calculated from Equations (5.6) and (5.10) where \( \alpha = 1 - 1/\tau \).

\[
SL_{k-1} + SL_k = 2\tau D \Rightarrow (1 + \alpha)SL_{k-1} = \frac{2}{1 - \alpha} D
\]

\[
SL_{k-1} = \frac{2}{1 - \alpha^2} D \quad \text{and} \quad SL_k = \frac{2\alpha}{1 - \alpha^2} D
\]

(5.16)

From Equations (5.11), (5.12) and (5.16):

\[
I_{k-1} = S - D - \frac{2}{1 - \alpha^2} D \quad \text{and} \quad I_k = S - 2D - \frac{2\alpha}{1 - \alpha^2} D
\]

(5.17)

To summarize, when \( D < S - s \leq 2D \) is satisfied, both supply line and inventory display period-2 oscillation with values given by Equations (5.16) and (5.17) respectively.

Simulation experiment: Consider parameters in Table 5.5. They satisfy \( D = 20 < (S - s) = 30 \leq 2D = 40 \) therefore period-2 oscillation must result as confirmed in Figure 5.2 and Table 5.6.

<table>
<thead>
<tr>
<th>Table 5.5. ((s,S)) parameters yielding P-2 oscillation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
</tr>
<tr>
<td>Values</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 5.6. ((s,S)) policy, P-2 oscillation, analytical and simulation results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
</tr>
<tr>
<td>Analytical results</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Simulation results</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
Figure 5.2. \((s, S)\) policy, first order continuous delay, P-2 oscillation

Period-3 Oscillation

If state of \(EI\) is assumed as Equation (5.18) at time \((k-1)\), using Equations (5.1) and (5.2) state of the system can be characterized by the following equations.

\[
EI_{k-1} > s \tag{5.18}
\]
\[
I_k = I_{k-1} + SL_{k-1}/\tau - D \tag{5.19}
\]
\[
SL_k = SL_{k-1} - SL_{k-1}/\tau \tag{5.20}
\]

If state of \(EI\) is assumed as Equation (5.21) at time \(k\), using Equations (5.1) and (5.2) state of the system can be characterized by the following equations.

\[
EI_k > s \tag{5.21}
\]
\[
I_{k+1} = I_k + SL_k/\tau - D \tag{5.22}
\]
\[
SL_{k+1} = SL_k - SL_k/\tau \tag{5.23}
\]
If state of $EI$ is assumed as Equation (5.24) at time $(k+1)$, using Equations (5.1) and (5.2) state of the system can be characterized by the following equations.

$$EI_{k+1} \leq s$$  \hspace{1cm} (5.24)

$$I_{k+2} = I_{k+1} + SL_{k+1}/\tau - D$$  \hspace{1cm} (5.25)

$$SL_{k+2} = SL_{k+1} + S - I_{k+1} - SL_{k+1} - SL_{k+1}/\tau$$  \hspace{1cm} (5.26)

Under what condition does the system display period-3 oscillation ($I_{k+2} = I_{k-1}$ and $SL_{k+2} = SL_{k-1}$)?

From Equations (5.19), (5.22) and (5.25):

$$I_{k+2} = I_{k-1} + \frac{SL_{k-1}}{\tau} - D + \frac{SL_k}{\tau} - D + \frac{SL_{k+1}}{\tau} - D = I_{k-1} + \frac{SL_{k-1} + SL_k + SL_{k+1}}{\tau} - 3D$$

$$I_{k-1} = I_{k-1} + \frac{SL_{k-1} + SL_k + SL_{k+1}}{\tau} - 3D$$

$$SL_{k-1} + SL_k + SL_{k+1} = 3\tau D$$  \hspace{1cm} (5.27)

From Equations (5.20), (5.23), (5.26) and (5.27):

$$SL_{k+2} = SL_{k-1} - \frac{SL_{k-1}}{\tau} - \frac{SL_k}{\tau} + S - I_{k+1} - SL_{k+1} - \frac{SL_{k+1}}{\tau}$$

$$SL_{k-1} = SL_{k-1} - \frac{SL_{k-1} + SL_k + SL_{k+1}}{\tau} + S - I_{k+1} - SL_{k+1}$$

$$EI_{k+1} = S - 3D$$  \hspace{1cm} (5.28)

If Equations (5.22) and (5.23) are added, $EI_{k+1} = EI_k - D$. Using $EI_{k+1} = S - 3D$:

$$S - 3D = EI_k - D \Rightarrow EI_k = S - 2D$$  \hspace{1cm} (5.29)

If Equations (5.19) and (5.20) are added, $EI_k = EI_{k-1} - D$. Using $EI_k = S - 2D$:
\[ S - 2D = EI_{k-1} - D \Rightarrow EI_{k-1} = S - D \]  \hspace{1cm} (5.30)

Finally three equations, Equations (5.28), (5.29) and (5.30), that are in the form of behavior of effective inventory, Equations (5.24), (5.21) and (5.18), are obtained. Considering these six equations we find the condition for the system to display period-3 oscillation.

Using Equations (5.18) and (5.30):

\[ \begin{align*}
EI_{k-1} > s \\
EI_{k-1} = S - D
\end{align*} \Rightarrow (S - D) > s \Rightarrow D < (S - s) \]  \hspace{1cm} (5.31)

Using Equations (5.21) and (5.29):

\[ \begin{align*}
EI_k > s \\
EI_k = S - 2D
\end{align*} \Rightarrow (S - 2D) > s \Rightarrow 2D < (S - s) \]  \hspace{1cm} (5.32)

Using Equations (5.24) and (5.28):

\[ \begin{align*}
EI_{k+1} \leq s \\
EI_{k+1} = S - 3D
\end{align*} \Rightarrow (S - 3D) \leq s \Rightarrow (S - s) \leq 3D \]  \hspace{1cm} (5.33)

From Equations (5.32) and (5.33) (Note that Equation (5.31) is redundant due to Equation (5.32)):

\[ 2D < S - s \leq 3D \]  \hspace{1cm} (5.34)

If the relationship between \( s \) and \( S \) satisfies Equation (5.34) then the system displays period-3 oscillation.

Supply line oscillation values can be calculated from Equations (5.20), (5.23) and (5.27) where \( \alpha = 1 - 1/\tau \).
\[ SL_{k-1} + SL_k + SL_{k+1} = 3D \Rightarrow (1 + \alpha + \alpha^2)SL_{k-1} = \frac{3}{1-\alpha}D \]

\[ SL_{k-1} = \frac{3}{1-\alpha^3}D \text{ and } SL_k = \frac{3\alpha}{1-\alpha^3}D \text{ and } SL_{k+1} = \frac{3\alpha^2}{1-\alpha^3}D \] (5.35)

From Equations (5.28), (5.29), (5.30) and (5.35):

\[ I_{k-1} = S - D - \frac{3}{1-\alpha^3}D, \quad I_k = S - 2D - \frac{3\alpha}{1-\alpha^3}D \text{ and } I_{k+1} = S - 3D - \frac{3\alpha^2}{1-\alpha^3}D \] (5.36)

To summarize, when \(2D < S - s \leq 3D\) is satisfied, both supply line and inventory display period-3 oscillation with values given by Equation (5.35) and (5.36) respectively.

Simulation experiment: Consider parameters in Table 5.7. They satisfy \(2D = 40 < (S - s) = 50 \leq 3D = 60\) therefore period-3 oscillation must result as confirmed in Figure 5.3 and Table 5.8.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>D</th>
<th>Q</th>
<th>( \tau )</th>
<th>s</th>
<th>S</th>
<th>( t_0 )</th>
<th>SL_{10}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>20</td>
<td>50</td>
<td>4</td>
<td>100</td>
<td>150</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5.7. \((s, S)\) parameters yielding P-3 oscillation

Figure 5.3. \((s, S)\) policy, first order continuous delay, P-3 oscillation
Table 5.8. \((s, S)\) policy, P-3 oscillation, analytical and simulation results

<table>
<thead>
<tr>
<th></th>
<th>Time</th>
<th>43</th>
<th>44</th>
<th>45</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Analytical results</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Supply line</td>
<td></td>
<td>103,78378</td>
<td>77,83784</td>
<td>58,37838</td>
</tr>
<tr>
<td>Inventory</td>
<td></td>
<td>26,21622</td>
<td>32,16216</td>
<td>31,62162</td>
</tr>
<tr>
<td><strong>Simulation results</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Supply line</td>
<td></td>
<td>103,78</td>
<td>77,84</td>
<td>58,38</td>
</tr>
<tr>
<td>Inventory</td>
<td></td>
<td>26,22</td>
<td>32,16</td>
<td>31,62</td>
</tr>
</tbody>
</table>

**Period-\(n\) Oscillation**

Starting from time \((k - 1)\) exactly \((n - 1)\) times effective inventory is greater than order point \(s\). At the \(n^{th}\) time from \((k - 1)\) (i.e. at time \((k + n - 2)\)) effective inventory drops to order point \(s\) or lower.

Using Equations (5.1) and (5.2):

\[
EI_{k-1} > s \Rightarrow \begin{cases} \frac{I_k}{\tau} = I_{k-1} + \frac{SL_{k-1}}{\tau} - D \\ SL_k = SL_{k-1} - \frac{SL_{k-1}}{\tau} \end{cases}
\]

\[
EI_k > s \Rightarrow \begin{cases} I_{k+1} = I_k + \frac{SL_k}{\tau} - D \\ SL_{k+1} = SL_k - \frac{SL_k}{\tau} \end{cases}
\]

\[
\vdots
\]

\[
EI_{k+n-3} > s \Rightarrow \begin{cases} I_{k+n-2} = I_{k+n-3} + \frac{SL_{k+n-3}}{\tau} - D \\ SL_{k+n-2} = SL_{k+n-3} - \frac{SL_{k+n-3}}{\tau} \end{cases}
\]

\[
EI_{k+n-2} \leq s \Rightarrow \begin{cases} I_{k+n-1} = I_{k+n-2} + \frac{SL_{k+n-2}}{\tau} - D \\ SL_{k+n-1} = SL_{k+n-2} + S - I_{k+n-2} - \frac{SL_{k+n-2}}{\tau} \end{cases}
\]

Under what condition does the system display period-\(n\) oscillation \((I_{k+n-1} = I_{k-1}\) and \(SL_{k+n-1} = SL_{k-1}\))?

If we write inventory equations of Equations (5.37) and (5.38) in each other, we obtain:
\[ I_{k+n-1} = I_{k-1} + \sum_{j=k-1}^{k+n-2} \left( \frac{SL_j}{\tau} - D \right) = I_{k-1} + \frac{1}{\tau} \sum_{j=k-1}^{k+n-2} SL_j - nD \Rightarrow I_{k-1} = I_{k-1} + \frac{1}{\tau} \sum_{j=k-1}^{k+n-2} SL_j - nD \]

\[ \sum_{j=k-1}^{k+n-2} SL_j = n\tau D \quad (5.39) \]

If we write supply line equations of Equations (5.37) and (5.38) in each other and use Equation (5.39), we obtain:

\[ SL_{k+n-1} = SL_{k-1} - \sum_{j=k-1}^{k+n-2} \frac{SL_j}{\tau} + S - I_{k+n-2} - SL_{k+n-2} \Rightarrow SL_{k-1} = SL_{k-1} - nD + S - EI_{k+n-2} \]

\[ EI_{k+n-2} = S - nD \quad (5.40) \]

If equation pairs in Equation (5.37) from time point \((k-1)\) to \((k+n-2)\) are added, the following relationship is obtained between effective inventories at successive time points.

\[ EI_k = EI_{k-1} - D \]
\[ EI_{k+1} = EI_k - D \]
\[ \vdots \]
\[ EI_{k+n-2} = EI_{k+n-3} - D \quad (5.41) \]

Using Equations (5.40) and (5.41) and going in the reverse the following effective inventory values are obtained.

\[ EI_{k+n-2} = S - nD \]
\[ EI_{k+n-3} = S - (n-1)D \]
\[ \vdots \]
\[ EI_k = S - 2D \]
\[ EI_{k-1} = S - D \quad (5.42) \]

Finally \(n\) equations, Equation (5.42), that are in the form of behavior of effective inventory are obtained. Considering Equation (5.42) together with behavior of effective
inventory in Equations (5.37) and (5.38), we find the condition for the system to display period-\(n\) oscillation.

\[
\begin{align*}
  k + n - 2 : S - nD \leq s & \Rightarrow S - s \leq nD \\
  k + n - 3 : S - (n-1)D > s & \Rightarrow S - s > (n-1)D \\
  \vdots \\
  k : S - 2D > s & \Rightarrow S - s > 2D \\
  k - 1 : S - D > s & \Rightarrow S - s > D
\end{align*}
\] (5.43)

In Equation (5.43), only the first two equations are effective. (Remaining ones are redundant due to the second equation). Using these two effective equations:

\[
(n-1)D < (S - s) \leq nD 
\] (5.44)

If the relationship between \(s\) and \(S\) satisfies Equation (5.44) then the system displays period-\(n\) oscillation.

Supply line oscillation values can be calculated from Equations (5.37) and (5.39).

\[
\sum_{j=k-1}^{k+n-2} SL_j = n\tau D \Rightarrow SL_{k-1} + SL_k + \cdots + SL_{k+n-2} = n\tau D
\]

\[
(1 + \alpha + \cdots + \alpha^{n-1})SL_{k-1} = \frac{nD}{1-\alpha} \Rightarrow SL_{k-1} = \frac{n}{1-\alpha^n}D
\] (5.45)

where \(\alpha = 1 - 1/\tau\).

Since \(SL_j = \alpha SL_{j-1}\) for \(j = k, k+1, \ldots, k+n-2\) supply line values are simply calculated from Equation (5.45).

Lastly, after subtracting supply line values from effective inventory values, Equation (5.42), inventory values are obtained.
\[ SL_{k-1} = \frac{n}{1-\alpha^n} D \]
\[ SL_k = \frac{n\alpha}{1-\alpha^n} D \]
\[ SL_{k+1} = \frac{n\alpha^2}{1-\alpha^n} D \]
\[ \vdots \]
\[ SL_{k+n-2} = \frac{n\alpha^{n-1}}{1-\alpha^n} D \]
\[ I_{k-1} = S - D - \frac{n}{1-\alpha^n} D \]
\[ I_k = S - 2D - \frac{n\alpha}{1-\alpha^n} D \]
\[ I_{k+1} = S - 3D - \frac{n\alpha^2}{1-\alpha^n} D \]
\[ \vdots \]
\[ I_{k+n-2} = S - nD - \frac{n\alpha^{n-1}}{1-\alpha^n} D \]

(5.46)

To summarize, both supply line and inventory display period-\(n\) oscillation with values given by Equation (5.46) where \(n\) satisfies \((n-1)D < (S-s) \leq nD\).

5.1.3. Simulation Experiments

Example 1: Consider parameters in Table 5.9. They satisfy \(4D = 80 < (S-s) = 90 \leq 5D = 100\) therefore period-5 oscillation must result as confirmed in Figure 5.4 and Table 5.10. Figure 5.4 is a phase map. Phase maps take one variable on one axis and another one on the other axis. Periodic behaviors are seen better in phase maps.

Table 5.9. \((s,S)\) parameters yielding P-5 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>D</th>
<th>Q</th>
<th>(\tau)</th>
<th>s</th>
<th>S</th>
<th>(I_0)</th>
<th>SL_0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>20</td>
<td>90</td>
<td>5</td>
<td>120</td>
<td>210</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5.10. \((s,S)\) policy, P-5 oscillation, analytical and simulation results

<table>
<thead>
<tr>
<th></th>
<th>Time</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>226</td>
<td>227</td>
<td>228</td>
<td>229</td>
<td>230</td>
<td></td>
</tr>
<tr>
<td>Analytical results</td>
<td>Supply line</td>
<td>148,7387</td>
<td>118,991</td>
<td>95,1928</td>
<td>76,1542</td>
<td>60,9234</td>
</tr>
<tr>
<td></td>
<td>Inventory</td>
<td>41,2613</td>
<td>51,0090</td>
<td>54,8072</td>
<td>53,8458</td>
<td>49,0766</td>
</tr>
<tr>
<td>Simulation results</td>
<td>Supply line</td>
<td>148,74</td>
<td>118,99</td>
<td>95,19</td>
<td>76,15</td>
<td>60,92</td>
</tr>
<tr>
<td></td>
<td>Inventory</td>
<td>41,26</td>
<td>51,01</td>
<td>54,81</td>
<td>53,85</td>
<td>49,08</td>
</tr>
</tbody>
</table>
Figure 5.4. $(s, S)$ policy, first order continuous delay, P-5 oscillation phase map

Example 2: Consider parameters in Table 5.11. They satisfy $7D = 140 < (S - s) = 150 \leq 8D = 160$ therefore period-8 oscillation must result as confirmed in Figure 5.5 and Table 5.12.

Table 5.11. $(s, S)$ parameters yielding P-8 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>D</th>
<th>Q</th>
<th>t</th>
<th>s</th>
<th>S</th>
<th>$l_0$</th>
<th>$SL_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>20</td>
<td>150</td>
<td>4</td>
<td>100</td>
<td>250</td>
<td>50</td>
<td>50</td>
</tr>
</tbody>
</table>

Table 5.12. $(s, S)$ policy, P-8 oscillation, analytical and simulation results

<table>
<thead>
<tr>
<th></th>
<th>Time</th>
<th>225</th>
<th>226</th>
<th>227</th>
<th>228</th>
<th>229</th>
<th>230</th>
<th>231</th>
<th>232</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>results</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Supply line</td>
<td>117.8</td>
<td>133.35</td>
<td>100.013</td>
<td>75.009</td>
<td>56.257</td>
<td>42.193</td>
<td>31.645</td>
<td>23.733</td>
<td></td>
</tr>
<tr>
<td>Inventory</td>
<td>52.2</td>
<td>76.65</td>
<td>89.987</td>
<td>94.991</td>
<td>93.743</td>
<td>87.807</td>
<td>78.355</td>
<td>66.267</td>
<td></td>
</tr>
<tr>
<td>Simulation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>results</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Supply line</td>
<td>177.8</td>
<td>133.35</td>
<td>100.01</td>
<td>75.01</td>
<td>56.26</td>
<td>42.19</td>
<td>31.64</td>
<td>23.73</td>
<td></td>
</tr>
<tr>
<td>Inventory</td>
<td>52.2</td>
<td>76.65</td>
<td>89.99</td>
<td>94.99</td>
<td>93.74</td>
<td>87.81</td>
<td>78.36</td>
<td>66.27</td>
<td></td>
</tr>
</tbody>
</table>
Figure 5.5. \((s, S)\) policy, first order continuous delay, \(P-8\) oscillation phase map

Example 3: Consider parameters in Table 5.13. They satisfy \(3D = 15 < (S - s) = 20 \leq 4D = 20\) therefore period-4 oscillation must result as confirmed in Figure 5.6 and Table 5.14. In this example, the difference \((S - s)\) is at the border value.

Table 5.13. \((s, S)\) parameters yielding \(P-4\) oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>D</th>
<th>Q</th>
<th>k</th>
<th>s</th>
<th>S</th>
<th>I₀</th>
<th>SL₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>5</td>
<td>20</td>
<td>2</td>
<td>15</td>
<td>35</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5.14. \((s, S)\) policy, \(P-4\) oscillation, analytical and simulation results

<table>
<thead>
<tr>
<th></th>
<th>Time</th>
<th>25</th>
<th>26</th>
<th>27</th>
<th>28</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Analytical results</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Supply line</td>
<td></td>
<td>21,333</td>
<td>10,6667</td>
<td>5,3333</td>
<td>2,6667</td>
</tr>
<tr>
<td>Inventory</td>
<td></td>
<td>8,6667</td>
<td>14,3333</td>
<td>14,6667</td>
<td>12,3333</td>
</tr>
<tr>
<td><strong>Simulation results</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Supply line</td>
<td></td>
<td>21,33</td>
<td>10,67</td>
<td>5,33</td>
<td>2,67</td>
</tr>
<tr>
<td>Inventory</td>
<td></td>
<td>8,67</td>
<td>14,33</td>
<td>14,67</td>
<td>12,33</td>
</tr>
</tbody>
</table>
5.2. Order Point - Order Quantity \((s,Q)\) Policy

Order Point - Order Quantity rule is mathematically expressed as:

\[
O_k = Q \quad \text{if} \ (EI_k \leq s) \\
0 \quad \text{else}
\]  \hspace{1cm} (5.47)

5.2.1. Goal Seeking Behavior

Assume system parameters satisfy \(Q \leq D\). From Section 4.2.1, if \(Q \leq D \Rightarrow EI \leq s\)

\(\Rightarrow O = Q\) in the long run, system equations are as follows:

\[
I_k = I_{k-1} + SL_{k-1}/\tau - D \\
SL_k = SL_{k-1} + Q - SL_{k-1}/\tau
\]

Supply line equation does not depend on inventory. It is an independent first order constant coefficient nonhomogeneous difference equation. Since \(\tau \geq 1\), what behavior inventory exhibits, supply line exhibits goal seeking behavior.
\[ SL_k - SL_{k-1} = Q - SL_{k-1}/\tau = 0 \Rightarrow SL_c = \tau Q \]
\[ I_k - I_{k-1} = SL_{k-1}/\tau - D = (\tau Q)/\tau - D = Q - D \]

Therefore inventory has an equilibrium only when \( Q = D \). If \( Q \leq D \), inventory decreases by amount \((D - Q)\) at each time step in the long run while supply line exhibits the same behavior, goal seeking behavior.

To summarize, if \( Q < D \) is satisfied supply line exhibits goal seeking while inventory indefinitely decreases. If \( Q = D \) both stock variables exhibit goal seeking behavior.

Simulation experiments

Example 1: Consider parameters in Table 5.15. They satisfy \( Q = 1 < D = 10 \) therefore for supply line goal seeking behavior and for inventory ever decreasing behavior must result as confirmed in Figure 5.7.

Table 5.15. \((s, Q)\) parameters yielding goal seeking \(SL\), ever decreasing \(I\)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>D</th>
<th>Q</th>
<th>(\tau)</th>
<th>s</th>
<th>(I_0)</th>
<th>(SL_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>10</td>
<td>1</td>
<td>4</td>
<td>50</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

![Diagram](image-url)

Figure 5.7. \((s, Q)\) policy, goal seeking \(SL\) and ever decreasing \(I\)
Example 2: Consider parameters in Table 5.16. They satisfy $Q = D = 10$ therefore goal seeking behavior must result as confirmed in Figure 5.8.

Table 5.16. $(s, Q)$ parameters yielding goal seeking behavior

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$D$</th>
<th>$Q$</th>
<th>$t$</th>
<th>$S$</th>
<th>$I_0$</th>
<th>$SL_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>10</td>
<td>10</td>
<td>4</td>
<td>50</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 5.8. $(s, Q)$ policy, first order continuous delay, goal seeking behavior

5.2.2. Periodic Behaviors

From Section 4.2.1, for $Q = \frac{n}{l}D$, where $\frac{n}{l} > 1$, $n$ and $l$ are relatively prime, $EI$ displays period-$n$ oscillation in which at $l$ time points it drops to order point $s$ or lower.

Although it is not exactly known at which time points effective inventory values are at or lower than order point $s$, thus supply line equations are not exactly known, it is known that at $l$ time points effective inventory is less than or equal to $s$ in a period of length $n$. Therefore although individual supply line equations are not known, supply line at $(k + n - 1)$ can be written in terms of supply line at time $(k - 1)$. 
\[ EI_{k+1} \Rightarrow \begin{cases} I_k = I_{k-1} + SL_{k-1}/\tau - D \\ SL_k = ? \end{cases} \]

\[ EI_k \Rightarrow \begin{cases} I_{k+1} = I_k + SL_k/\tau - D \\ SL_{k+1} = ? \end{cases} \]

\[ EI_{k+1} \Rightarrow \begin{cases} I_{k+2} = I_{k+1} + SL_{k+1}/\tau - D \\ SL_{k+2} = ? \end{cases} \]

\[ \vdots \]

\[ EI_{k+n-2} \Rightarrow \begin{cases} I_{k+n-1} = I_{k+n-2} + SL_{k+n-2}/\tau - D \\ SL_{k+n-1} = ? \end{cases} \]  \tag{5.48}

To conclude that system displays period-\(n\) oscillation, one has to show \(Q = \frac{n}{l}D\) satisfies \(I_{k+n-1} = I_{k-1}\) and \(SL_{k+n-1} = SL_{k-1}\).

If we write inventory equations of Equation (5.48) in each other, we obtain:

\[ I_{k+n-1} = I_{k-1} + \sum_{j=k-1}^{k+n-2} \left( \frac{SL_j}{\tau} - D \right) \Rightarrow I_{k-1} = I_{k-1} + \frac{1}{\tau} \sum_{j=k-1}^{k+n-2} SL_j - nD \]

\[ \sum_{j=k-1}^{k+n-2} SL_j = n\tau D \]  \tag{5.49}

If we write supply line equations of Equation (5.48) in each other and use Equation (5.49), we obtain:

\[ SL_{k+n-1} = SL_{k-1} - \sum_{j=k-1}^{k+n-2} \frac{SL_j}{\tau} + lQ \Rightarrow SL_{k-1} = SL_{k-1} - nD + lQ \Rightarrow Q = (n/l)D \]

To summarize, if \(Q = \frac{n}{l}D\) where \(\frac{n}{l} > 1\), \(n\) and \(l\) are relatively prime system (both supply line and inventory stocks) displays period-\(n\) oscillation.

About irrational relationships between \(Q\) and \(D\), the discussions made in Section 4.2.1 is valid. To summarize any irrational number can be approximated to a certain extent
by a rational number. After approximating the irrational number \( Q/D \) by a rational number whose numerator and denominator are relatively prime both stock variables display period-numerator oscillation (the better the approximation, the better the results).

To summarize, if \( Q/D > 1 \) and rational, system displays period-numerator oscillation after simplifying \( Q/D \) until numerator and denominator are relatively prime. If \( Q/D > 1 \) and irrational, system displays period-numerator oscillation after approximating \( Q/D \) by a rational number whose numerator and denominator are relatively prime.

5.2.3. Simulation Experiments

Example 1: Consider parameters in Table 5.17. They satisfy \( Q/D = 5/3 \) therefore period-5 oscillation must result as confirmed in Figure 5.9.

Table 5.17. (\( s, Q \)) parameters yielding P-5 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>D</th>
<th>Q=(5/3)D</th>
<th>( \tau )</th>
<th>( s )</th>
<th>( I_0 )</th>
<th>( SL_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>9</td>
<td>15</td>
<td>2</td>
<td>27</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 5.9. (\( s, Q \)) policy, first order continuous delay, P-5 oscillation
Example 2: Consider parameters in Table 5.18. They satisfy $Q/D = 4$ therefore period-4 oscillation must result as confirmed in Figure 5.10.

Table 5.18. $(s, Q)$ parameters yielding P-4 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>D</th>
<th>Q=4D</th>
<th>$\tau$</th>
<th>$s$</th>
<th>$l_0$</th>
<th>SL$_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>10,21</td>
<td>40,84</td>
<td>2.8</td>
<td>38,798</td>
<td>5</td>
<td>20</td>
</tr>
</tbody>
</table>

Figure 5.10. $(s, Q)$ policy, first order continuous delay, P-4 oscillation phase map

Example 3: Consider parameters in Table 5.19. By trial and error $Q = 15/4$ can be used to represent $Q = \sqrt{14}D$ where $\text{%error} = \%0.223$. Therefore period-15 oscillation must result as confirmed in Table 5.20 and Figure 5.11.

Table 5.19. $(s, Q)$ parameters yielding P-15 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>D</th>
<th>Q= $\sqrt{14}$ D</th>
<th>$\tau$</th>
<th>$s$</th>
<th>$l_0$</th>
<th>SL$_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>1</td>
<td>$\sqrt{14}$</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>20</td>
</tr>
</tbody>
</table>
Figure 5.11. \((s, Q)\) policy, first order continuous delay, P-15 oscillation

Table 5.20. \((s, Q)\) policy, P-15 oscillation, 2 period simulation results

<table>
<thead>
<tr>
<th>Time</th>
<th>84</th>
<th>85</th>
<th>86</th>
<th>87</th>
<th>88</th>
<th>89</th>
<th>90</th>
<th>91</th>
<th>92</th>
<th>93</th>
<th>94</th>
<th>95</th>
<th>96</th>
<th>97</th>
<th>98</th>
</tr>
</thead>
<tbody>
<tr>
<td>Supply Line</td>
<td>2.28</td>
<td>1.52</td>
<td>4.75</td>
<td>3.17</td>
<td>2.11</td>
<td>1.41</td>
<td>4.68</td>
<td>3.12</td>
<td>2.08</td>
<td>1.39</td>
<td>4.67</td>
<td>3.11</td>
<td>2.07</td>
<td>5.12</td>
<td>3.42</td>
</tr>
<tr>
<td>Inventory</td>
<td>2.33</td>
<td>2.09</td>
<td>1.6</td>
<td>2.18</td>
<td>2.24</td>
<td>1.94</td>
<td>1.41</td>
<td>1.97</td>
<td>2.01</td>
<td>1.7</td>
<td>1.17</td>
<td>1.72</td>
<td>1.76</td>
<td>1.45</td>
<td>2.16</td>
</tr>
<tr>
<td>Time</td>
<td>84</td>
<td>85</td>
<td>86</td>
<td>87</td>
<td>88</td>
<td>89</td>
<td>90</td>
<td>91</td>
<td>92</td>
<td>93</td>
<td>94</td>
<td>95</td>
<td>96</td>
<td>97</td>
<td>98</td>
</tr>
<tr>
<td>Supply Line</td>
<td>2.28</td>
<td>1.52</td>
<td>4.75</td>
<td>3.17</td>
<td>2.11</td>
<td>1.41</td>
<td>4.68</td>
<td>3.12</td>
<td>2.08</td>
<td>1.39</td>
<td>4.67</td>
<td>3.11</td>
<td>2.07</td>
<td>5.12</td>
<td>3.42</td>
</tr>
<tr>
<td>Inventory</td>
<td>2.33</td>
<td>2.09</td>
<td>1.6</td>
<td>2.18</td>
<td>2.24</td>
<td>1.94</td>
<td>1.41</td>
<td>1.97</td>
<td>2.01</td>
<td>1.7</td>
<td>1.17</td>
<td>1.72</td>
<td>1.76</td>
<td>1.45</td>
<td>2.16</td>
</tr>
</tbody>
</table>

5.3. Review Period - Order Up to Level \((R, S)\) Policy

Review Period – Order Up to Level rule is mathematically expressed as:

\[
O_k = S - EI_k \quad \text{if} \mod(k, R) = 0 \\
0 \quad \text{else}
\]

(5.50)
5.3.1. Goal Seeking Behavior

Assume system parameters satisfy $R = 1$. When $R = 1$, for any $k \mod (k, 1) = 0 \Rightarrow O_k = S - EI_k$ always. From Section 5.1.1, if $(S - s) \leq D$ then $EI \leq s$ and $O_k = S - EI_k$ always. Therefore $(R, S)$ policy with $R = 1$ is exactly equivalent to $(s, S)$ policy with $(S - s) \leq D$. As a result, when $R = 1$, both inventory and supply line have stable equilibrium points given by $I_e = S - (\tau + 1)D$ and $SL_e = \tau D$ respectively.

5.3.2. Periodic Behaviors

In Section 4.3.1, it is shown that $R$ determines behavior of $EI$. When dynamics of $EI$ is known, system equations can be easily written using Equations (5.1) and (5.50).

$R$ is assumed to be a positive integer greater than 1. From difference calculus it is known that $R$ simultaneous order-$R$ difference equations produce period-$R$ oscillation if absolute value of each eigen value of each of the $R$ equations is less than or equal to 1. Therefore first, equation pairs starting from $(Rk + 2)$ to $(Rk + 2R)$ are written. Then inventory and supply line at point $(Rk + R + 1)$ in terms of inventory and supply line at $(Rk + 1)$, inventory and supply line at point $(Rk + R + 2)$ in terms of inventory and supply line at $(Rk + 2)$, ..., inventory and supply line at point $(Rk + 2R)$ in terms of inventory and supply line at $(Rk + R)$ are obtained.

For all points other than $(Rk + R + 1)$, equation pairs are:

\[
I_{j+1} = I_j + SL_j / \tau - D
\]
\[
SL_{j+1} = SL_j - \frac{SL_j}{\tau} = \alpha SL_j
\]
\[
\Rightarrow EI_{j+1} = EI_j - D \quad j = \{Rk + 1, \ldots, Rk + 2R - 1\} - \{Rk + R\}
\]

(5.51)

At time $(Rk + R + 1)$, equation pair changes since $\mod(Rk + R, R) = 0$ (order $O_{Rk+R} = S - EI_{Rk+R}$).
\[ I_{Rk+R+1} = I_{Rk+R} + SL_{Rk+R} / \tau - D \]
\[ SL_{Rk+R+1} = SL_{Rk+R} + S - I_{Rk+R} - SL_{Rk+R} \frac{SL_{Rk+R}}{\tau} = S - I_{Rk+R} - \frac{SL_{Rk+R}}{\tau} \Rightarrow EI_{Rk+R+1} = S - D \]

From Equations (5.51) and (5.52), for time points greater than \((Rk + R)\) effective inventory values are:

\[ EI_{Rk+R+1} = S - D \]
\[ EI_{Rk+R+2} = EI_{Rk+R+1} - D = S - 2D \]
\[ EI_{Rk+R+3} = EI_{Rk+R+2} - D = S - 3D \]
\[ \vdots \]
\[ EI_{Rk+2R} = EI_{Rk+2R-1} - D = S - RD \]  

(5.53)

Note that at time point \((Rk + R + 1)\) equations are changing. Inventory and supply line at a general \((Rk + R + l)\) in terms of inventory and supply line at \((Rk + l)\) are derived. Then by plugging \(l=1,2,\ldots,n\) others are obtained. We start with inventory, using Equations (5.51) and (5.52):

\[ I_{Rk+R+l} = I_{Rk+l} + \frac{\sum_{j=Rk+l}^{Rk+R+l-1} SL_j}{\tau} - RD = I_{Rk+l} + \frac{\sum_{j=Rk+l}^{Rk+R+l-1} SL_j}{\tau} - RD \]

\[ I_{Rk+R+l} = I_{Rk+l} + \frac{1+\alpha+\ldots+\alpha^{R-l}}{\tau} SL_{Rk+l} + (1+\alpha+\ldots+\alpha^{R-l}) SL_{Rk+R+l} - RD \]

\[ I_{Rk+R+l} = I_{Rk+l} + \frac{1-\alpha^{R-l+1}}{\tau} SL_{Rk+l} + \frac{1-\alpha^{R-l}}{\tau} SL_{Rk+R+l} - RD \]

\[ I_{Rk+R+l} = I_{Rk+l} + (1-\alpha^{R-l+1}) SL_{Rk+l} + (1-\alpha^{R-l}) SL_{Rk+R+l} - RD \]

\[ I_{Rk+R+l} = I_{Rk+l} + (1-\alpha^{R-l+1}) SL_{Rk+l} + (1-\alpha^{R-l}) (S - I_{Rk+l} - \frac{SL_{Rk+R}}{\tau}) - RD \]

\[ I_{Rk+R+l} = I_{Rk+l} - (1-\alpha^{R-l}) I_{Rk+l} + (1-\alpha^{R-l+1}) SL_{Rk+l} - (1-\alpha^{R-l}) \frac{SL_{Rk+R}}{\tau} + (1-\alpha^{R-l}) S - RD \]

\[ I_{Rk+R+l} = I_{Rk+l} - (1-\alpha^{R-l}) I_{Rk+l} + (1-\alpha^{R-l+1}) SL_{Rk+l} - (1-\alpha^{R-l})(1-\alpha)\alpha^{R-l} SL_{Rk+l} + (1-\alpha^{R-l}) S - RD \]
\[ I_{Rk+R+l} = I_{Rk+l} - (1 - \alpha^{l-1})I_{Rk+R} + (1 - \alpha^{R-l} + \alpha^{R-l} - \alpha^{R})SL_{Rk+l} + (1 - \alpha^{l-1})S - RD \]  
(5.54)

\[ I_{Rk+R} \] must be written it in terms of \( I_{Rk+l} \) and \( SL_{Rk+l} \).

\[ I_{Rk+R} = \frac{\sum_{j=Rk+l}^{Rk+R-1} SL_j}{\tau} - (R - l)D = I_{Rk+l} + \frac{(1 + \alpha + \ldots + \alpha^{R-l-1})SL_{Rk+l} - (R - l)D}{\tau} \]

\[ I_{Rk+R} = I_{Rk+l} + \frac{1 - \alpha^{R-l}}{\tau} SL_{Rk+l} - (R - l)D \]

\[ I_{Rk+R} = I_{Rk+l} + (1 - \alpha^{R-l})SL_{Rk+l} - (R - l)D \]  
(5.55)

Plug Equation (5.55) into Equation (5.54).

\[ I_{Rk+R+l} = I_{Rk+l} - (1 - \alpha^{l-1})I_{Rk+R} + (1 - \alpha^{R-l} + \alpha^{R-l} - \alpha^{R})SL_{Rk+l} + (1 - \alpha^{l-1})S - RD \]

\[ I_{Rk+R+l} = I_{Rk+l} - (1 - \alpha^{l-1})(I_{Rk+l} + (1 - \alpha^{R-l})SL_{Rk+l} - (R - l)D) + (1 - \alpha^{R-l} + \alpha^{R-l} - \alpha^{R})SL_{Rk+l} + (1 - \alpha^{l-1})S - RD \]

\[ I_{Rk+R+l} = \alpha^{l-1}I_{Rk+l} + (1 - \alpha^{R-l} + \alpha^{R-l} - \alpha^{R} - (1 - \alpha^{l-1})(1 - \alpha^{R-l})SL_{Rk+l} + (1 - \alpha^{l-1})S - RD + (1 - \alpha^{l-1})(R - l)D \]

\[ I_{Rk+R+l} = \alpha^{l-1}I_{Rk+l} + (\alpha^{l-1} - \alpha^{R})SL_{Rk+l} + S - lD - \alpha^{l-1}(S + (R - l)D) \]  
(5.56)

Now we obtain supply line, using Equations (5.51) and (5.52):

\[
\begin{align*}
SL_{Rk+R+l-1} & = \alpha SL_{Rk+R+l-1} \\
\vdots & \\
SL_{Rk+R+2} & = \alpha SL_{Rk+R+1}
\end{align*}
\]

\[ \Rightarrow SL_{Rk+R+l} = \alpha^{l-1}SL_{Rk+R+l-1} \]  
(5.57)

\[ SL_{Rk+R+l} = S - I_{Rk+R} - SL_{Rk+R}/\tau \]  
(5.58)
\[
\begin{aligned}
SL_{rk+r} &= \alpha SL_{rk+r-1} \\
SL_{rk+r-1} &= \alpha SL_{rk+r-2} \\
&\vdots \\
SL_{rk+i+1} &= \alpha SL_{rk+i} \\
\end{aligned}
\implies SL_{rk+r} = \alpha^{r-i} SL_{rk+i}
\] (5.59)

Now, consider Equations (5.57), (5.58) and (5.59) together.

\[
SL_{rk+r+1} = \alpha^{l-1}(S - I_{rk+r} \cdot \frac{SL_{rk+r}}{\tau}) = -\alpha^{l-1}I_{rk+r} - \alpha^{l-1}(1 - \alpha)\alpha^{r-1}SL_{rk+i} + \alpha^{l-1}S
\] (5.60)

Plug Equation (5.55) in Equation (5.60):

\[
SL_{rk+r+1} = -\alpha^{l-1}(I_{rk+r} + (1 - \alpha^{r-1})SL_{rk+i} - (R - l)D) - \alpha^{l-1}(1 - \alpha)\alpha^{r-1}SL_{rk+i} + \alpha^{l-1}S
\]

\[
SL_{rk+r+1} = -\alpha^{l-1}I_{rk+r} + (\alpha^{r-1} + \alpha^{r-1} - (1 - \alpha)\alpha^{r-1})SL_{rk+i} + \alpha^{l-1}S + \alpha^{l-1}(R - l)D
\]

\[
SL_{rk+r+1} = -\alpha^{l-1}I_{rk+i} - (\alpha^{r-1} - \alpha^{r})SL_{rk+i} + \alpha^{l-1}(S + (R - l)D)
\] (5.61)

Both \(I_{rk+r+1}\) and \(SL_{rk+r+1}\) are written in terms of \(I_{rk+i}\) and \(SL_{rk+i}\), Equations (5.56) and (5.61). A quick check is \(I_{rk+r+1} + SL_{rk+r+1} = S - lD\) which is consistent with Equation (5.53).

\[
\begin{pmatrix}
I_{rk+r+1} \\
SL_{rk+r+1}
\end{pmatrix} =
\begin{pmatrix}
\alpha^{l-1} & (\alpha^{l-1} - \alpha^{r}) \\
-\alpha^{l-1} & -(\alpha^{l-1} - \alpha^{r})
\end{pmatrix}
\begin{pmatrix}
I_{rk+i} \\
SL_{rk+i}
\end{pmatrix} +
\begin{pmatrix}
S - lD - \alpha^{l-1}(S + (R - l)D) \\
\alpha^{l-1}(S + (R - l)D)
\end{pmatrix}
\] (5.62)

Now, say we have a matrix of the form \(B = \begin{pmatrix} a & b \\ -a & -b \end{pmatrix}\).

\[
B = \begin{pmatrix} a & b \\ -a & -b \end{pmatrix} \implies |B - \lambda I| = \begin{vmatrix} a - \lambda & b \\ -a & -b - \lambda \end{vmatrix} = \lambda(\lambda + (b - a)) = 0
\]

\[
\begin{cases}
\text{either } \lambda = 0 \\
or \quad \lambda = a - b
\end{cases}
\] (5.63)
Apply Equation (5.63) to Equation (5.62), \[\begin{cases}
\text{either } \lambda = 0 \\
or \quad \lambda = \alpha^{l^1} - (\alpha^{l^1} - \alpha^{R}) = \alpha^{R}
\end{cases}\]

For any value of \( l \), \( \lambda = 0 \) or \( \lambda = \alpha^{R} \) and since \(|\alpha| = |1 - 1/\tau| \leq 1\), roots of the characteristic equation are in the unit circle, so the system is stable (absolute value eigenvalue of each of \( R \) equations is less than or equal to 1).

We are interested in the long term behavior of the system, so we do not try to find the homogenous solution since it vanishes as time increases. (Remember homogenous solution consists of power of roots of characteristic equation which are shown to be smaller than 1). Since nonhomogenous part is composed of constant terms, try a constant vector for inventory and supply line values at \((Rk + R + l)\) in the long term.

\[
k = \begin{pmatrix}
I_{Rk+R+l} \\
SL_{Rk+R+l}
\end{pmatrix}
\]

In general assume \( k \) and \( a \) are 2 by 1 constant vectors.

\[
k = Ak + a \Rightarrow k(I - A) = a \Rightarrow k = (I - A)^{-1}a = \begin{pmatrix} 1-a & -b \\ a & 1+b \end{pmatrix}^{-1}a = \frac{1}{|I - A|} \begin{pmatrix} 1+b & b \\ -a & 1-a \end{pmatrix}a
\]

\[
k = \frac{1}{1+b-a} \begin{pmatrix} 1+b & b \\ -a & 1-a \end{pmatrix}a
\]

(5.64)

Apply Equation (5.64) to Equation (5.62).

\[
k = \frac{1}{1 + (\alpha^{l^1} - \alpha^{R}) - \alpha^{l^1}} \times \begin{pmatrix}
1 + (\alpha^{l^1} - \alpha^{R}) & (\alpha^{l^1} - \alpha^{R}) \\
-\alpha^{l^1} & 1 - \alpha^{l^1}
\end{pmatrix}
\begin{pmatrix}
S - ID - \alpha^{l^1}(S + (R - I)D) \\
\alpha^{l^1} (S + (R - I)D)
\end{pmatrix}
\]

\[
SL_{Rk+R+l} = \frac{R\alpha^{l^1}}{1 - \alpha^{R}}D
\]

(5.65)
If we add Equations (5.56) and (5.61), \( I_{Rk + R + l} + SL_{Rk + R + l} = S - l D \). Subtract Equation (5.65).

\[
I_{Rk + R + l} = S - l D - \frac{R \alpha^{l-1}}{1 - \alpha} D
\]

(5.66)

So both inventory and supply line values at \((Rk + R + l)\) are found. Now, plug \(l = 1, 2, \ldots, n\) into Equations (5.56), (5.61), (5.65) and (5.66).

\[
\begin{align*}
I_{Rk + R + 2} &= \alpha I_{Rk + 2} + (\alpha - \alpha^R) SL_{Rk + 2} + S - 2D - \alpha(S + (R - 2)D) \\
SL_{Rk + R + 2} &= -\alpha I_{Rk + 2} - (\alpha - \alpha^R) SL_{Rk + 2} + \alpha(S + (R - 2)D)
\end{align*}
\]

\[
\Rightarrow \begin{cases}
\text{either } \lambda = 0 \\
\text{or } \lambda = \alpha^R \Rightarrow SL_{Rk + R + 2} = \frac{R \alpha}{1 - \alpha^R} D
\end{cases}
\]

\[
\begin{align*}
I_{Rk + R + 3} &= \alpha^2 I_{Rk + 3} + (\alpha^2 - \alpha^R) SL_{Rk + 3} + S - lD - \alpha^{l-1}(S + (R - l)D) \\
SL_{Rk + R + 3} &= -\alpha^2 I_{Rk + 3} - (\alpha^2 - \alpha^R) SL_{Rk + 3} + \alpha^2(S + (R - l)D)
\end{align*}
\]

\[
\Rightarrow \begin{cases}
\text{either } \lambda = 0 \\
\text{or } \lambda = \alpha^R \Rightarrow SL_{Rk + R + 3} = \frac{R \alpha^{l-1}}{1 - \alpha^R} D
\end{cases}
\]

\[
\vdots
\]

\[
\begin{align*}
I_{Rk + R + 2R} &= \alpha^{2R-1} I_{Rk + R} + (\alpha^{2R-1} - \alpha^R) SL_{Rk + R} + S - RD - \alpha^{2R-1}S \\
SL_{Rk + R + 2R} &= -\alpha^{2R-1} I_{Rk + R} - (\alpha^{2R-1} - \alpha^R) SL_{Rk + R} + \alpha^{2R-1}(S + (R - l)D)
\end{align*}
\]

\[
\Rightarrow \begin{cases}
\text{either } \lambda = 0 \\
\text{or } \lambda = \alpha^R \Rightarrow SL_{Rk + R + 2R} = \frac{R \alpha^{2R-1}}{1 - \alpha^R} D
\end{cases}
\]

Thus, supply line values are found from \((Rk + R + 1)\) to \((Rk + 2R)\). Effective inventory values are also known from Equation (5.53). Inventory values can be obtained easily by subtracting supply line values from effective inventory values.
\[ SL_{Rk+R+1} = \frac{R}{1-\alpha^R} D \quad I_{Rk+R+1} = S - D - \frac{R}{1-\alpha^R} D \]
\[ SL_{Rk+R+2} = \frac{R\alpha}{1-\alpha^R} D \quad I_{Rk+R+2} = S - 2D - \frac{R\alpha}{1-\alpha^R} D \]
\[ SL_{Rk+R+3} = \frac{R\alpha^2}{1-\alpha^R} D \quad I_{Rk+R+3} = S - 3D - \frac{R\alpha^2}{1-\alpha^R} D \]
\[ \vdots \quad \vdots \]
\[ SL_{Rk+2R} = \frac{R\alpha^{R-1}}{1-\alpha^R} D \quad I_{Rk+2R} = S - RD - \frac{R\alpha^{R-1}}{1-\alpha^R} D \]

If \( R \) is not an integer, the discussion made in Section 4.3.1 is valid. However, as stated, to choose a noninteger \( R \) is not meaningful when discrete time step, \( k \), is specified as a small enough step.

To summarize, for an integer \( R > 1 \), both supply line and inventory display period-\( R \) oscillation with values given by Equation (5.67).

### 5.3.3. Simulation Experiments

Example 1: Consider parameters in Table 5.21. They satisfy \( R = 5 \) therefore period-5 oscillation must result as confirmed in Figure 5.12 and Table 5.22.

**Table 5.21. \((R,S)\) parameters yielding P-5 oscillation**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>R</th>
<th>D</th>
<th>r</th>
<th>S</th>
<th>I₀</th>
<th>SL₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>5</td>
<td>20</td>
<td>5</td>
<td>220</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 5.22. \((R,S)\) policy, P-5 oscillation, analytical and simulation results**

<table>
<thead>
<tr>
<th>Time</th>
<th>86</th>
<th>87</th>
<th>88</th>
<th>89</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical results</td>
<td>Supply line</td>
<td>148,7387</td>
<td>118,9910</td>
<td>95,1928</td>
<td>76,1542</td>
</tr>
<tr>
<td></td>
<td>Inventory</td>
<td>51,2613</td>
<td>61,009</td>
<td>64,8072</td>
<td>63,8458</td>
</tr>
<tr>
<td>Simulation results</td>
<td>Supply line</td>
<td>148,74</td>
<td>118,99</td>
<td>95,19</td>
<td>76,15</td>
</tr>
<tr>
<td></td>
<td>Inventory</td>
<td>51,26</td>
<td>61,01</td>
<td>64,81</td>
<td>63,85</td>
</tr>
</tbody>
</table>
Figure 5.12. ($R, S$) policy, first order continuous delay, P-5 oscillation phase map

Example 2: Consider parameters in Table 5.23. They satisfy $R = 8$ therefore period-8 oscillation must result as confirmed in Figure 5.13.

Table 5.23. ($R, S$) parameters yielding P-8 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$R$</th>
<th>$D$</th>
<th>$\tau$</th>
<th>$S$</th>
<th>$I_0$</th>
<th>$SL_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>8</td>
<td>20</td>
<td>4</td>
<td>260</td>
<td>30</td>
<td>25</td>
</tr>
</tbody>
</table>

Figure 5.13. ($R, S$) policy, first order continuous delay, P-8 oscillation phase map
5.4. \((R, s, S)\) Policy

\((R, s, S)\) rule is mathematically expressed as:

\[
O_k = S - EI_k \quad \text{if} \ (\mod(k, R) = 0 \text{ and } EI_k \leq s) \\
0 \quad \text{else}
\] (5.68)

\((R, s, S)\) policy is a combination of \((s, S)\) and \((R, S)\). Since enough experience is built on \((s, S)\) and \((R, S)\) policies, we present a rather intuitive discussion using what we learnt while studying \((R, S)\) and \((s, S)\).

If \(R = 1\) then first condition in if statement in Equation (5.68) is always satisfied and \((R, s, S)\) completely turns into \((s, S)\) and all the results in \((s, S)\) are valid.

If \((S - s) \leq D \) (or \((S - D) \leq s\)) then in the long run effective inventory is always less than or equal to order point \(s\) since \(S - D\) is the maximum value effective inventory can take. Therefore second condition in the if statement in Equation (5.68) is always satisfied. As a result, \((R, s, S)\) completely turns into \((R, S)\), and all the results in \((R, S)\) are valid.

\(R > 1\) and/or \((S - s) > D\) is a new case. System under \((R, S)\) policy displays period-\(R\) oscillation. However in \((R, s, S)\), effective inventory is also checked against order point \(s\) at time points multiples of \(R\) so it is intuitive that \((R, s, S)\) displays oscillatory behavior in a multiple of \(R\).

5.4.1. Goal Seeking Behavior

\((R, s, S)\) with \(R = 1\) is exactly equivalent to \((s, S)\) and from Section 5.1.1, \((s, S)\) exhibits goal seeking behavior for \((S - s) \leq D\). Therefore, if \(R = 1\) and \((S - s) \leq D\), system exhibits goal seeking behavior.
5.4.2. Periodic Behaviors

Proof is similar to \((R,S)\), please refer to Appendix B.

Result of this proof is that \((R,s,S)\) displays period-\(nR\) oscillation where \(n\) is the smallest integer greater than the ratio \((S-s)/(RD)\) with the following points.

\[
\begin{align*}
SL_{Rk+nR+1} &= \frac{n}{1-\alpha^{nR}} RD \quad & I_{Rk+nR+1} &= S - D - \frac{n}{1-\alpha^{nR}} RD \\
SL_{Rk+nR+2} &= \frac{n\alpha}{1-\alpha^{nR}} RD \quad & I_{Rk+nR+2} &= S - 2D - \frac{n\alpha}{1-\alpha^{nR}} RD \\
SL_{Rk+nR+3} &= \frac{n\alpha^2}{1-\alpha^{nR}} RD \quad & I_{Rk+nR+3} &= S - 3D - \frac{n\alpha^2}{1-\alpha^{nR}} RD \\
\vdots & \quad & \vdots \\
SL_{Rk+nR+nR} &= \frac{n\alpha^{nR-1}}{1-\alpha^{nR}} RD \quad & I_{Rk+nR+nR} &= S - nRD - \frac{n\alpha^{nR-1}}{1-\alpha^{nR}} RD
\end{align*}
\]  

(5.69)

If \(R\) is not an integer, the discussion made in Section 4.3.1 is entirely valid here. However, as stated, to choose a noninteger \(R\) is not meaningful when discrete time step, \(k\), is specified as a small enough step.

To summarize, if \(R = 1\) and \((S-s) \leq D\), \((R,s,S)\) exhibits goal seeking behavior. If integer \(R > 1\) and/or \((S-s) > D\), system displays period-\(nR\) oscillation where \(n\) is the smallest integer greater than the ratio \((S-s)/(RD)\) with the points given in Equation (5.69).

5.4.3. Simulation Experiments

Example 1: Consider parameters in Table 5.24. They satisfy \(R = 1\) and \((S-s) = 10 \leq D = 20\) therefore goal seeking behavior must result with \(SL_e = \tau D = 80\) and \(I_e = S - (\tau + 1)D = 30\) as confirmed in Figure 5.14.
Table 5.24. \((R,s,S)\) parameters yielding goal seeking behavior

<table>
<thead>
<tr>
<th>Parameters</th>
<th>R</th>
<th>D</th>
<th>(\tau)</th>
<th>s</th>
<th>S</th>
<th>(I_0)</th>
<th>SL_0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>1</td>
<td>20</td>
<td>4</td>
<td>120</td>
<td>130</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

![Graph showing supply line and inventory over time](image)

Figure 5.14.\((R,s,S)\) policy, first order continuous delay, goal seeking behavior

Example 2: Consider parameters in Table 5.25. They satisfy \(R = 1 \Rightarrow \) exactly \((s,S) \Rightarrow 4D = 80 < (S-s) = 90 \leq 5D = 100\) therefore period-5 oscillation must result as confirmed in Figure 5.15 and Table 5.26.

Table 5.25. \((R,s,S)\) parameters yielding P-5 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>R</th>
<th>D</th>
<th>(\tau)</th>
<th>s</th>
<th>S</th>
<th>(I_0)</th>
<th>SL_0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>1</td>
<td>20</td>
<td>5</td>
<td>140</td>
<td>230</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5.26. \((R,s,S)\) policy, P-5 oscillation, analytical and simulation results

<table>
<thead>
<tr>
<th>Time</th>
<th>Analytical results</th>
<th>Simulation results</th>
</tr>
</thead>
<tbody>
<tr>
<td>226</td>
<td>Supply line</td>
<td>148,7387</td>
</tr>
<tr>
<td>227</td>
<td>Supply line</td>
<td>118,9910</td>
</tr>
<tr>
<td>228</td>
<td>Inventory</td>
<td>61,2613</td>
</tr>
<tr>
<td>229</td>
<td>Inventory</td>
<td>71,0090</td>
</tr>
<tr>
<td>230</td>
<td></td>
<td>74,8072</td>
</tr>
<tr>
<td></td>
<td></td>
<td>73,8458</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60,9234</td>
</tr>
<tr>
<td></td>
<td></td>
<td>69,0766</td>
</tr>
<tr>
<td></td>
<td></td>
<td>69,08</td>
</tr>
</tbody>
</table>
Figure 5.15. \((R,s,S)\) policy, first order continuous delay, P-5 oscillation

Example 3: Consider parameters in Table 5.27. They satisfy \((S-s) = 15 \leq D = 20 \Rightarrow \text{exactly} (R,S)\) since \(R = 4\) period-4 oscillation must result as confirmed in Figure 5.16.

Table 5.27. \((R,s,S)\) parameters with first order delay yielding P-4 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>R</th>
<th>D</th>
<th>τ</th>
<th>s</th>
<th>S</th>
<th>I₀</th>
<th>SL₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>4</td>
<td>20</td>
<td>2</td>
<td>140</td>
<td>155</td>
<td>0</td>
<td>50</td>
</tr>
</tbody>
</table>

Figure 5.16. \((R,s,S)\) policy, first order continuous delay, P-4 oscillation
Example 4: Consider parameters in Table 5.28. They satisfy \( R = 3 \) and 
\( (S - s) = 25 > D = 5 \Rightarrow (S - s)/RD = 25/15 \leq n \Rightarrow n = 2 \) therefore period-6 oscillation must result as confirmed in Figure 5.17.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( R )</th>
<th>( D )</th>
<th>( s )</th>
<th>( S )</th>
<th>( I_0 )</th>
<th>( SL_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>30</td>
<td>55</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5.28. \((R,s,S)\) parameters yielding P-6 oscillation

![Supply Line v. Inventory](image)

Figure 5.17. \((R,s,S)\) policy, first order continuous delay, P-6 oscillation

5.5. Chapter Conclusion

\((s,S)\) policy exhibits two different types of behaviors which are completely determined according to the relationship between \( s, S \) and \( D \). If \( (S - s) \leq D \) supply line and inventory exhibit goal seeking behavior. If \( (n - 1)D < (S - s) \leq nD, \quad n \geq 2 \) they display period-\( n \) oscillation. Table 5.29 summarizes how system behaves under what conditions and gives corresponding supply line and inventory value formulas.

Behavior of \((s,Q)\) is fully defined by the relationship between \( Q \) and \( D \). If \( Q < D \), \( SL \) exhibits goal seeking and inventory indefinitely decreases. If \( Q = D \), both stocks exhibit goal seeking behavior. If \( Q/D > 1 \) and rational, system displays period-numerator
oscillation after simplifying $Q/D$ until numerator and denominator are relatively prime. If $Q/D > 1$ and irrational, the system displays period-numerator oscillation after approximating $Q/D$ with a rational number whose numerator and denominator are relatively prime. Table 5.30 summarizes how system behaves under what conditions.

Behavior of $(R, S)$ policy is completely determined by $R$. If $R = 1$, the system exhibits goal seeking behavior. If integer $R > 1$ system displays period-$R$ oscillation. Table 5.31 summarizes how system behaves under what conditions and gives corresponding supply line and inventory value formulas. Behavior of $(R, s, S)$ policy is determined by the relationship between $R, s, S$ and $D$. If $R = 1$ and $(S - s) \leq D$, the system exhibits goal seeking behavior. If integer $R > 1$ and/or $(S - D) > s$, the system displays period-$nR$ oscillation where $n$ is the smallest integer greater than or equal to the ratio $(S - s)/(RD)$. Table 5.32 summarizes how system behaves under what conditions and gives corresponding supply line and inventory value formulas.

Table 5.29. $SL$ and $I$ behavior and formulas under $(s, S)$ policy with first order delay

<table>
<thead>
<tr>
<th>Condition</th>
<th>Behavior</th>
<th>Supply line and inventory formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(S - s) \leq D$</td>
<td>Goal seeking</td>
<td>$SL = \tau D I_\ast = S - (\tau + 1)D$</td>
</tr>
<tr>
<td>$D &lt; (S - s)$</td>
<td>Period-2 oscillation</td>
<td>$SL_{i+1} = \frac{2}{1 - \alpha^i} D, SL_i = \frac{2\alpha}{1 - \alpha^i} D$</td>
</tr>
<tr>
<td>$(S - s) \leq 2D$</td>
<td></td>
<td>$I_{i+1} = S - D - \frac{2}{1 - \alpha^i} D, I_i = S - 2D - \frac{2\alpha}{1 - \alpha^i} D$</td>
</tr>
<tr>
<td>$2D &lt; (S - s)$</td>
<td>Period-3 oscillation</td>
<td>$SL_{i+1} = \frac{3}{1 - \alpha^i} D, SL_i = \frac{3\alpha}{1 - \alpha^i} D, SL_{i+1} = \frac{3\alpha^2}{1 - \alpha^i} D$</td>
</tr>
<tr>
<td>$(S - s) \leq 3D$</td>
<td></td>
<td>$I_{i+1} = S - D - \frac{3}{1 - \alpha^i} D, I_i = S - 2D - \frac{3\alpha}{1 - \alpha^i} D, I_{i+1} = S - 3D - \frac{3\alpha^2}{1 - \alpha^i} D$</td>
</tr>
<tr>
<td>$(n - 1)D &lt; (S - s)$</td>
<td>Period-$n$ oscillation</td>
<td>$SL_{i+1} = \frac{n}{1 - \alpha^i} D, SL_i = \frac{n\alpha}{1 - \alpha^i} D, \ldots, SL_{i+n-1} = \frac{n\alpha^{n-1}}{1 - \alpha^i} D$</td>
</tr>
<tr>
<td>$(S - s) \leq nD$</td>
<td></td>
<td>$I_{i+1} = S - D - \frac{n}{1 - \alpha^i} D, I_i = S - 2D - \frac{n\alpha}{1 - \alpha^i} D, \ldots, I_{i+n-1} = S - nD - \frac{n\alpha^{n-1}}{1 - \alpha^i} D$</td>
</tr>
</tbody>
</table>
Table 5.30. SL and I behavior under \((s,Q)\) policy with first order continuous delay

<table>
<thead>
<tr>
<th>Condition</th>
<th>Behavior of supply line and inventory</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q &lt; D)</td>
<td>Supply line goal seeking, inventory ever decreasing</td>
</tr>
<tr>
<td>(Q = D)</td>
<td>Goal seeking</td>
</tr>
<tr>
<td>Rational (Q/D &gt; 1)</td>
<td>Period-numerator oscillation</td>
</tr>
<tr>
<td>(simplify (Q/D) until numerator and denominator are relatively prime)</td>
<td></td>
</tr>
<tr>
<td>Irrational (Q/D &gt; 1)</td>
<td>Approximately period-numerator oscillation</td>
</tr>
<tr>
<td>(approximate with a rational number whose numerator and denominator are relatively prime)</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.31. SL and I behavior and formulas under \((R,S)\) policy with first order delay

<table>
<thead>
<tr>
<th>Condition</th>
<th>Behavior</th>
<th>Supply line and inventory formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R = 1)</td>
<td>Goal seeking</td>
<td>(SL_t = rD I_t = S - (r + 1)D)</td>
</tr>
<tr>
<td>(R = 2)</td>
<td>Period-2 oscillation</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(SL_{n+1} = \frac{2}{1 - \alpha^2} D, SL_{n+1} = \frac{2\alpha}{1 - \alpha^2} D)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(I_{n+1} = S - D - \frac{2}{1 - \alpha^2} D, I_{n+1} = S - 2D - \frac{2\alpha}{1 - \alpha^2} D)</td>
<td></td>
</tr>
<tr>
<td>(R = 3)</td>
<td>Period-3 oscillation</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(SL_{n+1} = \frac{3}{1 - \alpha^3} D, SL_{n+1} = \frac{3\alpha}{1 - \alpha^3} D, SL_{n+1} = \frac{3\alpha^2}{1 - \alpha^3} D)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(I_{n+1} = S - D - \frac{3}{1 - \alpha^3} D, I_{n+1} = S - 2D - \frac{3\alpha}{1 - \alpha^3} D, I_{n+1} = S - 3D - \frac{3\alpha^2}{1 - \alpha^3} D)</td>
<td></td>
</tr>
<tr>
<td>(\cdots)</td>
<td>(\cdots)</td>
<td>(\cdots)</td>
</tr>
<tr>
<td>Integer (R &gt; 1)</td>
<td>Period-(R) oscillation</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(SL_{n+1} = \frac{R}{1 - \alpha^R} D, SL_{n+1} = \frac{R\alpha}{1 - \alpha^R} D, \cdots, SL_{n+1} = \frac{R\alpha^{R-1}}{1 - \alpha^R} D)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(I_{n+1} = S - D - \frac{R}{1 - \alpha^R} D, I_{n+1} = S - 2D - \frac{R\alpha}{1 - \alpha^R} D, \cdots, I_{n+1} = S - RD - \frac{R\alpha^{R-1}}{1 - \alpha^R} D)</td>
<td></td>
</tr>
</tbody>
</table>
Table 5.32. \( SL \) and \( I \) formulas under \((R,s,S)\) policy with first order delay

<table>
<thead>
<tr>
<th>Condition</th>
<th>Behavior</th>
<th>Supply line and inventory formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R = 1 ) and ( (S-s) \leq D )</td>
<td>Goal seeking</td>
<td>( SL_e = \tau D I_e = S - (\tau + 1)D )</td>
</tr>
<tr>
<td>( R &gt; 1 ) and/or ( (S-s) &gt; D )</td>
<td>Period-( nR ) oscillation where ( n ) is the smallest integer greater than or equal to ( (S-s)/(RD) )</td>
<td>( SL_{n-1} = \frac{n}{1-\alpha^R} RD, SL_{n+1} = \frac{n\alpha}{1-\alpha^R} RD, \ldots, SL_{n+\infty} = \frac{n\alpha^{\infty}}{1-\alpha^R} RD ) ( I_{n-1} = S - D - \frac{n}{1-\alpha^R} RD, I_{n+1} = S - 2D - \frac{n\alpha}{1-\alpha^R} RD, \ldots, I_{n+\infty} = S - nRD - \frac{n\alpha^{\infty}}{1-\alpha^R} R )</td>
</tr>
</tbody>
</table>

Infact, supply line and inventory formulas of \((s,S),(R,S)\) and \((R,s,S)\) policies are very similar; Table 5.33 summarizes supply line and inventory formulas for these three policies where \( P \) is the period (in \((s,S)\) \( P = n \), in \((R,S)\) \( P = R \) and in \((R,s,S)\) \( P = nR \)).

Table 5.33. \( SL \) and \( I \) formulas under \((s,S),(R,S)\) and \((R,s,S)\) policies

<table>
<thead>
<tr>
<th>Behavior</th>
<th>Supply line and inventory formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goal seeking</td>
<td>( SL_e = \tau D I_e = S - (\tau + 1)D )</td>
</tr>
<tr>
<td>Period-P oscillation</td>
<td>( SL_{k-1} = \frac{PD}{1-\alpha^p}, SL_k = \frac{\alpha PD}{1-\alpha^p}, \ldots, SL_{k+p-2} = \frac{\alpha^{p-1}PD}{1-\alpha^p} ) ( I_{k-1} = S - D - \frac{PD}{1-\alpha^p}, I_k = S - 2D - \frac{\alpha PD}{1-\alpha^p}, \ldots, I_{k+p-2} = S - PD - \frac{\alpha^{p-1}PD}{1-\alpha^p} )</td>
</tr>
</tbody>
</table>
6. INVENTORY DYNAMICS FORMULAS WITH HIGHER ORDER DELAYS

In this chapter, dynamics of inventory with $M^{th}$ order continuous delay are analyzed under different policies.

From Chapter 3, it is known that Stock Acquisition sector for all inventory policies is same. Therefore, independent of the inventory policy, it is represented by the following set of equations for an $M^{th}$ order continuous delay system.

\[ I_k = I_{k-1} + \frac{SL^M}{\tau_M} - D \]

\[ SL^m_k = SL^m_{k-1} + \frac{SL^{m-1}}{\tau_{m-1}} - \frac{SL^m}{\tau_m}, m = 2, \ldots, M \quad (6.1) \]

\[ SL^1_k = SL^1_{k-1} + O_{k-1} - \frac{SL^1}{\tau_1} \]

Order equation $O_{k-1}$ is different for each policy. For all inventory policies with $M^{th}$ order continuous delay structure, delays $\tau_m \geq 1, m = 1, 2, \ldots, M$ otherwise outflows of supply line stocks exceed level of stocks (since the model is discrete) which is illogical.

Techniques in Chapter 5, for analyzing each inventory policy can be extended such that $M^{th}$ order continuous delay inventory policies are analyzed with a little difficulty and more mathematics. However, we prefer to carry out the analysis in another way for two reasons. Firstly the analysis technique explained in this chapter is also valid for first order continuous delay systems so it is a verification of Chapter 5. Secondly and more importantly since the analysis technique is using small atomic structures independent of inventory policy considered, although more complex systems are being analyzed, analysis becomes much simpler. Results are even not restricted to continuous delay systems, since atomic structures are the foundations of analysis. These results are also used in Chapter 7 in mixed delay systems.
6.1. Analysis of Atomic Structures

System analyzed in Sections 6.1.1 and 6.1.2 resemble the supply line structures and those analyzed in Sections 6.1.3 and 6.1.4 resemble the inventory structure. The application and similarity of these atomic structures to inventory policies are elaborated in Section 6.2.

6.1.1. Constant Inflow-Proportional Outflow Atomic Structure

Stock flow diagram of constant inflow proportional outflow atomic structure is shown in Figure 6.1.

Figure 6.1. Stock flow diagram of constant inflow proportional outflow structure

This system is represented by the following equation where \( i_k \) represents constant inflow.

\[
SL_k = SL_{k-1} + i_{k-1} - SL_{k-1}/\tau = \alpha SL_{k-1} + i_{k-1}
\]  

(6.2)

where \( \alpha = 1 - 1/\tau \).

Equilibrium point is given by Equation (6.3) and the system is stable since \(|\alpha| = |1 - 1/\tau| < 1\).

\[
SL_k - SL_{k-1} = i - SL_{k-1}/\tau = 0 \Rightarrow SL_e = \tau i
\]

(6.3)
As a result, if inflow eventually becomes constant, stock variable exhibits goal seeking behavior with the equilibrium point given by Equation (6.3) and outflow also exhibits goal seeking behavior since it is proportional to stock variable.

The application of Equation (6.3) to \( M^{th} \) order continuous delay systems: If we consider first supply line in Stock Acquisition sector constant inflow \( i \) is order \( O_k \), if we consider any other supply line stock \( m \), constant inflow \( i \) is outflow of supply line stock \( m-1 \). Therefore, supply line and inventory formulas for \((s,S),(R,S)\) and \((R,s,S)\) policies are obtained as in Table 6.11 for goal seeking behavior case.

6.1.2. Periodic Inflow-Proportional Outflow Atomic Structure

Stock flow diagram of periodic inflow proportional outflow atomic structure is shown in Figure 6.2.

![Figure 6.2. Stock flow diagram of periodic inflow proportional outflow structure](image)

This system is represented by the following equation where inflow \( i_k \) is periodic.

\[
SL_k = SL_{k-1} + i_{k-1} - SL_{k-1}/\tau = \alpha SL_{k-1} + i_{k-1}
\]  

(6.4)

where \( \alpha = 1 - 1/\tau \).

From difference calculus, it is known that \( n \) simultaneous order-\( n \) equations produce period-\( n \) oscillation if absolute value of each eigen value of each of the \( n \) equations is less than or equal to 1. For a general \( R=n \), we try to write these \( n \)
simultaneous order-\(n\) equations for our periodic inflow proportional outflow atomic structure; \(SL\) at point \((nk+1)\) in terms of \(SL\) at \((nk+1-n)\), \(SL\) at point \((nk+2)\) in terms of \(SL\) at \((nk+2-n)\), …, and \(SL\) at point \((nk+n)\) in terms of \(SL\) at \(nk\). To accomplish this \(SL\) at a general \((nk+l)\) point in terms of \(SL\) at \((nk+l-n)\) are derived. Then by plugging \(l=1,2,\ldots,n\), others are obtained.

Equation (6.5) defines periodic inflow proportional outflow system during \(n\) consecutive time points.

\[
SL_{j+1} = \alpha SL_j + i_j, \quad j = nk+l-n, \ldots, nk+l-1
\]  

(6.5)

If we write equations of Equation (6.5) from time \(nk+l-n\) to \(nk+l-1\) in each other, we obtain \(SL\) at \((nk+l)\) in terms of \(SL\) at \((nk+l-n)\).

\[
\begin{align*}
SL_{nk+l} &= \alpha SL_{nk+l-1} + i_{nk+l-1} = \alpha(\alpha SL_{nk+l-2} + i_{nk+l-2}) + i_{nk+l-1} = \alpha^2 SL_{nk+l-2} + \alpha i_{nk+l-2} + i_{nk+l-1} \\
SL_{nk+l} &= \alpha^3 SL_{nk+l-3} + \alpha^2 i_{nk+l-3} + \alpha i_{nk+l-2} + i_{nk+l-1} \\
&\vdots \\
SL_{nk+l} &= \alpha^n S_{nk+l-n} + \alpha^{n-1} i_{nk+l-n} + \alpha^{n-2} i_{nk+l-n+1} + \cdots + \alpha i_{nk+l-2} + i_{nk+l-1} \\
SL_{nk+l} &= \alpha^n SL_{nk+l-n} + \sum_{j=1}^{n} \alpha^{j-1} i_{nk+l-j}
\end{align*}
\]  

(6.6)

For any value of \(l\), \(\lambda = \alpha^n\) and since \(|\alpha| = |1-1/\tau| < 1\) the system is stable (absolute value of each of the \(n\) equations is less than or equal to 1).

We are interested in the long term behavior of the system, so we do not try to find the homogenous solution since it vanishes as time increases (remember \(|\alpha| < 1\)). Since nonhomogenous part is composed of constant terms, we try a constant for \(SL\) value, \(A\), in the long term and plug it in Equation (6.6).

\[
\begin{align*}
SL_{nk+l} &= \alpha^n SL_{nk+l-n} + \sum_{j=1}^{n} \alpha^{j-1} i_{nk+l-j} \Rightarrow A = \alpha^n A + \sum_{j=1}^{n} \alpha^{j-1} i_{nk+l-j}
\end{align*}
\]
\[(1 - \alpha^n)A = \sum_{j=1}^{n} \alpha^{j-1}i_{nk+l-j}\]

\[A = \frac{\sum_{j=1}^{n} \alpha^{j-1}i_{nk+l-j}}{1 - \alpha^n}\]

(6.7)

Plug \(l = 1, 2, \ldots, n\) into Equations (6.6) and (6.7).

\[SL_{nk+1} = \alpha^n SL_{nk+1-n} + \sum_{j=1}^{n} \alpha^{j-1}i_{nk+l-j} \Rightarrow \lambda = \alpha^n \Rightarrow SL_{nk+1} = \frac{\sum_{j=1}^{n} \alpha^{j-1}i_{nk+l-j}}{1 - \alpha^n}\]

\[SL_{nk+2} = \alpha^n SL_{nk+2-n} + \sum_{j=1}^{n} \alpha^{j-1}i_{nk+l+2-j} \Rightarrow \lambda = \alpha^n \Rightarrow SL_{nk+2} = \frac{\sum_{j=1}^{n} \alpha^{j-1}i_{nk+l+2-j}}{1 - \alpha^n}\]

\[\vdots\]

\[SL_{nk+l} = \alpha^n SL_{nk+l-n} + \sum_{j=1}^{n} \alpha^{j-1}i_{nk+l-j} \Rightarrow \lambda = \alpha^n \Rightarrow SL_{nk+l} = \frac{\sum_{j=1}^{n} \alpha^{j-1}i_{nk+l-j}}{1 - \alpha^n}\]

\[\vdots\]

\[SL_{nk+n} = \alpha^n SL_{nk} + \sum_{j=1}^{n} \alpha^{j-1}i_{nk+n-j} \Rightarrow \lambda = \alpha^n \Rightarrow SL_{nk+n} = \frac{\sum_{j=1}^{n} \alpha^{j-1}i_{nk+n-j}}{1 - \alpha^n}\]

Therefore, if inflow eventually displays period-\(n\) oscillation, stock variable displays period-\(n\) oscillation with the points above and outflow displays period-\(n\) oscillation since it is proportional to stock.

The application of this section to \(M^{th}\) order continuous systems: If we consider first supply line in Stock Acquisition sector periodic inflow \(i\) is order \(O_k\), if we consider any other supply line stock \(m\), periodic inflow \(i\) is outflow of supply line stock \(m-1\). Therefore, supply line and inventory formulas for \((s,S)\), \((R,S)\) and \((R,s,S)\) policies are obtained as in Table 6.11 for periodic behavior case.
6.1.3. Constant Inflow-Constant Outflow Atomic Structure

Stock flow representation of constant inflow constant outflow atomic structure is shown in Figure 6.3.

Figure 6.3. Stock flow diagram of constant inflow constant outflow structure

This system is represented by the following equation where $i_k$ and $o_k$ represent constant inflow and outflow respectively.

$$SL_k = SL_{k-1} + i_{k-1} - o_{k-1} \quad (6.8)$$

which has an equilibrium and exhibits goal seeking behavior only when the following condition is satisfied.

$$i - o = 0 \quad (6.9)$$

6.1.4. Periodic Inflow-Constant Outflow Atomic Structure

Stock flow representation of periodic inflow constant outflow atomic structure is shown in Figure 6.4.

Figure 6.4. Stock flow diagram of periodic inflow constant outflow structure
This system is represented by the following equations where \( i_k \) and \( o_k \) represent periodic inflow and constant outflow respectively.

\[
SL_k = SL_{k-1} + i_{k-1} - o_{k-1}
\]  
(6.10)

In this situation Equation (6.6) turns into the following equation.

\[
SL_{nk+1} = SL_{nk+1-n} + \sum_{j=1}^{n} (i_{nk+1-j} - o)
\]

which has an equilibrium and displays period-\( n \) oscillation only when the following condition is satisfied.

\[
\sum_{j=1}^{n} (i_{nk+1-j} - o) = 0
\]  
(6.11)

6.2. Atomic Structures in Inventory Policies

From Section 4.5, it is known that for all inventory policies the order \( O_k \) displays either goal seeking or periodic oscillation and from Chapter 3, Stock Acquisition sector is exactly same in all inventory policies. Therefore the following discussion applies to all policies.

6.2.1. First Supply Line Structure

Behavior of \( SL^1 \), whose stock flow representation is shown in Figure 6.5, is defined by the following equation where \( \alpha_1 = 1 - 1/\tau_1 \).

\[
SL^1_k = SL^1_{k-1} + O_{k-1} - SL^1_{k-1}/\tau_1 = \alpha_1 SL^1_{k-1} + O_{k-1}
\]

which is in the form of Equations (6.2) and (6.4).
If order is constant then $SL^1$ and $f^1$ are goal seeking from Section 6.1.1. If order is periodic then $SL^1$ and $f^1$ are also periodic from Section 6.1.2.

6.2.2. Intermediate Supply Line Structures

Consider any intermediate supply line stock, without loss of generality say $SL^2$. Behavior of $SL^2$, whose stock flow representation is shown in Figure 6.6, is defined by the following equation where $f^1_{k-1} = \frac{SL^1_{k-1}}{\tau_1}$ and $\alpha_2 = 1 - 1/\tau_2$.

$$SL^2_k = SL^2_{k-1} + f^1_{k-1} - SL^2_{k-1}/\tau_2 = \alpha_2 SL^2_{k-1} + f^1_{k-1}$$

which is in the form of Equations (6.2) and (6.4).

If $SL^1$ and $f^1$ are goal seeking then $SL^2$ and $f^2$ are also goal seeking from Section 6.1.1. If $SL^1$ and $f^1$ are periodic then $SL^2$ and $f^2$ are also periodic from Section 6.1.2.
Same relationship is valid for any other intermediate supply line stock, behavior of $SL^m$ is defined by the following equation where $f^{m-1}_{k-1} = SL^m_{k-1} / \tau_m$ and $\alpha_m = 1 - 1 / \tau_m$.

$$SL^m_k = SL^m_{k-1} + f^{m-1}_{k-1} - SL^m_{k-1} / \tau_m = \alpha_m SL^m_{k-1} + f^{m-1}_{k-1}, \quad m = 2, 3, \ldots, M$$

which is in the form of Equations (6.2) and (6.4).

If $SL^{m-1}$ and $f^{m-1}$ are goal seeking then $SL^m$ and $f^m$ are also goal seeking from Section 6.1.1. If $SL^{m-1}$ and $f^{m-1}$ are periodic then $SL^m$ and $f^m$ are also periodic from Section 6.1.2.

To summarize, from Section 4.5, it is known that order displays just those two behavior types in any policy for any condition: either goal seeking or periodic oscillation. In Section 6.1.1 and Section 6.1.2, it is proven that if the outflow is proportional, both stock and outflow variables exhibit the same behavior type as inflow. Using these two results together with results of Section 6.2.1 and Section 6.2.2, both supply line stocks and outflows of them exhibit the same behavior type as order until, including, $f^M$. If order exhibits goal seeking behavior, all supply line stocks and all outflows of them exhibit goal seeking behavior; if order displays period-$n$ oscillation, all supply line stocks and all outflows of them also display period-$n$ oscillation.

### 6.2.3. Inventory Structure

Behavior of inventory, whose stock flow representation is shown in Figure 6.7, is defined by the following equation where $f^M_{k-1} = SL^M_{k-1} / \tau_M$.

$$I_k = I_{k-1} + f^M_{k-1} - D$$

which is in the form of Equations (6.8) and (6.10).
There exist two cases to be analyzed; in one case $SL^M$ and $f^M$ exhibit goal seeking behavior and in the other case they display periodic oscillation.

We start with the case where $SL^M$ and $f^M$ exhibit goal seeking behavior. From Equation (6.9), the following condition must be satisfied to conclude that inventory exhibits goal seeking behavior.

$$f_{i-1}^M - D = \frac{SL_{i-1}^M}{\tau_M} - D = 0$$

(6.12)

Goal seeking behavior of $SL^M$ and $f^M$ means order is constant since we showed that all supply line stocks with continuous delay structure exhibit the same behavior as order $O_k$ (conditions when constant order occurs is given in Section 4.5 for each policy). Value of this constant $O_k$ is proven to be $D$ in $(s,S),(R,S)$ and $(R,s,S)$ policies from Section 4.1.1, Section 4.3.1 and Appendix A respectively.

Using results of Section 6.1.1 and $O_k = D$, we obtain the following equilibrium values for supply line stocks.

$$SL^1_e = \tau_1 D, \; SL^2_e = \tau_2 D, \; \ldots, \; SL^M_e = \tau_M D$$

Since $SL^M_e = \tau_M D$, $SL^M_e / \tau_M - D = 0$ so Equation (6.12) is satisfied.

In $(s,Q)$ policy, constant order means $O_k = Q$ in the long run. From Section 4.5, in $(s,Q)$ policy, order $O_k$ is constant when $Q \leq D$. Equation (6.12) is not satisfied in $(s,Q)$
unless \( Q = D \). Using results of Section 6.1.1 and \( O_k = Q \), we obtain the following equilibrium values for supply line stocks and the following inventory equation.

\[
SL_1^e = \tau_1 D, \ SL_2^e = \tau_2 D, \ldots, \ SL_e^e = \tau_M D \\
I_k - I_{k-1} = SL_{k+1}^e / \tau_M - D = Q - D
\]

Therefore inventory has equilibrium and exhibits goal seeking behavior if \( Q = D \), otherwise (if \( Q < D \)) it indefinitely decreases.

The second case is where \( SL^M \) and \( f^M \) display periodic oscillation. From Equation (6.11) the following condition must be satisfied to conclude that inventory displays periodic oscillation.

\[
\sum_{j=1}^{n} (f_{k-j}^M - D) = \sum_{j=1}^{n} (SL_{k-j}^M / \tau_M - D) = 0 \tag{6.13}
\]

Since the case where \( SL^M \) and \( f^M \) display periodic oscillation is analyzed, order also displays periodic oscillation, without loss of generality lets say period- \( n \) (we showed that all supply line stocks with continuous delay structure exhibit same behavior as order). Although the order \( O_k \) displays periodic oscillation, thus equation of first supply line changes, other supply line equations and inventory equations are always same.

If we write equations of supply line \( M \) of Equation (6.1) into each other from time \( k - 1 \) to \( k + n - 1 \), we obtain:

\[
SL_{k+n-1}^M = SL_{k-1}^M + \sum_{j=k-1}^{k+n-2} (SL_{j-1}^M / \tau_{M-1} - SL_j^M / \tau_M)
\]

Since supply line \( M \) is known to display period- \( n \) oscillation, i.e. \( SL_{k+n-1}^M = SL_{k-1}^M \),

\[
\sum_{j=k-1}^{k+n-2} (SL_{j-1}^M / \tau_{M-1} - SL_j^M / \tau_M) = 0 \Rightarrow \sum_{j=k-1}^{k+n-2} SL_{j-1}^M / \tau_{M-1} = \sum_{j=k-1}^{k+n-2} SL_j^M / \tau_M.
\]
If we write equations of supply line $M - 1$ of Equation (6.1) into each other from time $k - 1$ to $k + n - 1$, we obtain:

\[
SL_{k+n-1}^{M-1} = SL_{k-1}^{M-1} + \sum_{j=k-1}^{k+n-2} \left( \frac{SL_j^{M-2}}{\tau_{M-2}} - \frac{SL_j^{M-1}}{\tau_{M-1}} \right)
\]

Since supply line $M - 1$ is known to make period-$n$ oscillation, i.e. $SL_{k+n-1}^{M-1} = SL_{k-1}^{M-1}$,

\[
\sum_{j=k-1}^{k+n-2} \left( \frac{SL_j^{M-2}}{\tau_{M-2}} - \frac{SL_j^{M-1}}{\tau_{M-1}} \right) = 0 \Rightarrow \sum_{j=k-1}^{k+n-2} \frac{SL_j^{M-2}}{\tau_{M-2}} = \sum_{j=k-1}^{k+n-2} \frac{SL_j^{M-1}}{\tau_{M-1}}.
\]

If we continue like this, at the end if we write equations of supply line 2 of Equation (6.1) into each other from time $k - 1$ to $k + n - 1$, we obtain:

\[
SL_{k+n-1}^{2} = SL_{k-1}^{2} + \sum_{j=k-1}^{k+n-2} \left( \frac{SL_j^{2}}{\tau_1} - \frac{SL_j^{2}}{\tau_2} \right)
\]

Since supply line 2 is known to make period-$n$ oscillation, i.e. $SL_{k+n-1}^{2} = SL_{k-1}^{2}$,

\[
\sum_{j=k-1}^{k+n-2} \left( \frac{SL_j^{2}}{\tau_1} - \frac{SL_j^{2}}{\tau_2} \right) = 0 \Rightarrow \sum_{j=k-1}^{k+n-2} \frac{SL_j^{1}}{\tau_1} = \sum_{j=k-1}^{k+n-2} \frac{SL_j^{2}}{\tau_2}.
\]

If we consider these consecutive results together, we obtain the following relationship between supply line values for all policies.

\[
\sum_{j=k-1}^{k+n-2} \frac{SL_j^{M}}{\tau_{M}} = \sum_{j=k-1}^{k+n-2} \frac{SL_j^{M-1}}{\tau_{M-1}} = \cdots = \sum_{j=k-1}^{k+n-2} \frac{SL_j^{2}}{\tau_2} = \sum_{j=k-1}^{k+n-2} \frac{SL_j^{1}}{\tau_1}
\]

Now, again we consider $(s, S), (R, S)$ and $(R, s, S)$ together and $(s, Q)$ separately. This is because orders dynamics of $(s, S), (R, S)$ and $(R, s, S)$ policies resemble to each other very much. In these three policies, order decision is taken only once in a period from Chapter 4 and $O_k = S - EI_k$, without loss of generality assume that point is $k + n - 2$. 
Then system equations (or stock acquisition sector equations) of these three policies and the relationship between effective inventory values are as follows:

\[
\begin{align*}
  I_{k} &= I_{k-1} + \frac{SL_{k-1}^{m}}{\tau_{M}} - D \\
  SL_{k}^{m} &= SL_{k-1}^{m} + \frac{SL_{k-1}^{m-1}}{\tau_{m-1}} - \frac{SL_{k-1}^{m}}{\tau_{m}}, m = 2, \ldots, M \\ 
  SL_{k}^{l} &= SL_{k-1}^{l} - \frac{SL_{k-1}^{l}}{\tau_{1}}
\end{align*}
\]

\[
\begin{align*}
  I_{k+1} &= I_{k} + \frac{SL_{k}^{m}}{\tau_{M}} - D \\
  SL_{k+1}^{m} &= SL_{k}^{m} + \frac{SL_{k}^{m-1}}{\tau_{m-1}} - \frac{SL_{k}^{m}}{\tau_{m}}, m = 2, \ldots, M \\ 
  SL_{k+1}^{l} &= SL_{k}^{l} - \frac{SL_{k}^{l}}{\tau_{1}}
\end{align*}
\]

\[
\begin{align*}
  I_{k+n-2} &= I_{k+n-3} + \frac{SL_{k+n-3}^{m}}{\tau_{M}} - D \\
  SL_{k+n-2}^{m} &= SL_{k+n-3}^{m} + \frac{SL_{k+n-3}^{m-1}}{\tau_{m-1}} - \frac{SL_{k+n-3}^{m}}{\tau_{m}}, m = 2, \ldots, M \\ 
  SL_{k+n-2}^{l} &= SL_{k+n-3}^{l} - \frac{SL_{k+n-3}^{l}}{\tau_{1}}
\end{align*}
\]

\[
\begin{align*}
  I_{k+n-1} &= I_{k+n-2} + \frac{SL_{k+n-2}^{m}}{\tau_{M}} - D \\
  SL_{k+n-1}^{m} &= SL_{k+n-2}^{m} + \frac{SL_{k+n-2}^{m-1}}{\tau_{m-1}} - \frac{SL_{k+n-2}^{m}}{\tau_{m}}, m = 2, \ldots, M \\ 
  SL_{k+n-1}^{l} &= SL_{k+n-2}^{l} + S - EI_{k+n-2} - \frac{SL_{k+n-3}^{l}}{\tau_{1}}
\end{align*}
\]

Since effective inventory is known to make period-\(n\) oscillation (same periodic behavior as order and supply lines), i.e. \(EI_{k+n-1} = EI_{k-1} = S - D\), using the relationship found above between effective inventory values, \(EI_{k+n-2} = S - nD\).
Now, if we write equations of supply line 1 of Equation (6.1) into each other from time \( k - 1 \) to \( k + n - 1 \) for \((s,S),(R,S)\) and \((R,s,S)\) policies where \( O_k = S - E_l_k \), we obtain:

\[
SL_{k+n-1}^1 = SL_{k-1}^1 + S - E_l_{k+n-2} - \sum_{j=k-1}^{k+n-2} SL_j^1 / \tau_1
\]

\[
SL_{k+n-1}^1 = SL_{k-1}^1 + S - (S - nD) - \sum_{j=k-1}^{k+n-2} SL_j^1 / \tau_1
\]

\[
SL_{k+n-1}^1 = SL_{k-1}^1 + nD - \sum_{j=k-1}^{k+n-2} SL_j^1 / \tau_1
\]

Since supply line 1 is known to make period-\( n \) oscillation, i.e. \( SL_{k+n-1}^1 = SL_{k-1}^1 \),

\[
\sum_{j=k-1}^{k+n-2} SL_j^1 / \tau_1 = nD.
\]

Lastly using the relationship between supply line stocks:

\[
\sum_{j=k-1}^{k+n-2} SL_j^M / \tau_M = nD \Rightarrow \sum_{j=k-1}^{k+n-2} (SL_j^M / \tau_M - D) = 0 \tag{6.14}
\]

Equations (6.13) and (6.14) are equivalent (since supply line \( M \) is known to display period-\( n \) oscillation) so Equation (6.13) is satisfied for \((s,S),(R,S)\) and \((R,s,S)\) policies.

We consider \((s,Q)\) policy separately since order decision is taken more than once in a period and order amount \( O_k = Q \). If we write equations of supply line 1 of Equation (6.1) into each other from time \( k - 1 \) to \( k + n - 1 \) for \((s,Q)\) policy, we obtain:

\[
SL_{k+n-1}^1 = SL_{k-1}^1 + lQ - \sum_{j=k-1}^{k+n-2} SL_j^1 / \tau_1
\]

\[
SL_{k+n-1}^1 = SL_{k-1}^1 + l \frac{n}{l} D - \sum_{j=k-1}^{k+n-2} SL_j^1 / \tau_1
\]

\[
SL_{k+n-1}^1 = SL_{k-1}^1 + nD - \sum_{j=k-1}^{k+n-2} SL_j^1 / \tau_1
\]
Since supply line 1 is known to make period- \( n \) oscillation, i.e. \( SL_{k+n-1}^L = SL_{k-1}^L \),
\[
\sum_{j=k-1}^{k+n-2} SL_j^L / \tau_i = nD.
\]
Lastly using the relationship between supply line values we again obtain Equation (6.14) which is equivalent to Equation (6.13), thus Equation (6.13) is satisfied for \((s, Q)\) policy.

As a result, when \( SL^M \) and \( f^M \) display periodic oscillation, inventory displays periodic oscillation in all four inventory policies.

6.3. **Inventory Dynamics of \( M^{th} \) Order Continuous Delay Systems**

We combine results of Sections 6.2 and 4.5.

\((s, S)\) policy exhibits goal seeking behavior if \((S - s) \leq D\) and it displays period-\( n \) oscillation when \((S - s) > D\) where \( n \) is the smallest integer greater than or equal to ratio \((S - s) / D\).

In \((s, Q)\) policy, all supply line stocks exhibit goal seeking behavior and inventory indefinitely decreases if \( Q < D \); they all exhibit goal seeking behavior if \( Q = D \). If \( Q / D > 1 \) and rational after simplifying \( Q / D \) until numerator and denominator are relatively prime they all display period-numerator oscillation. Lastly, if \( Q / D > 1 \) and irrational after approximating \( Q / D \) by a rational number such that numerator and denominator are relatively prime they both display period-numerator oscillation.

\((R, S)\) policy exhibits goal seeking behavior if \( R = 1 \). If integer \( R > 1 \), the system displays period-\( R \) oscillation.

\((R, s, S)\) policy exhibits goal seeking behavior if \( R = 1 \) and \((S - s) \leq D\). If integer \( R > 1 \) and/or \((S - s) > D\), the system displays period-\( nR \) where \( n \) is the smallest integer greater than or equal to the ratio \((S - s) / (RD)\).
6.4. Simulation Experiments

Example 1: Consider parameters in Table 6.1. They satisfy \((S - s)/D = 8/3\) therefore period-3 oscillation must result as confirmed in Figure 6.8.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>D</th>
<th>Q</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>s</th>
<th>S</th>
<th>(t_0)</th>
<th>(SL_1)</th>
<th>(SL_2)</th>
<th>(SL_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>(s,S)</td>
<td>2</td>
<td>3</td>
<td>8</td>
<td>5</td>
<td>2</td>
<td>24</td>
<td>32</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 6.8. \((s,S)\) policy, second order continuous delay, P-3 oscillation

Example 2: Consider parameters in Table 6.2. They satisfy \((S - s) = 1 < D = 2\) therefore goal seeking behavior must result as confirmed in Figure 6.9.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>D</th>
<th>Q</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>(\tau_3)</th>
<th>s</th>
<th>S</th>
<th>(t_0)</th>
<th>(SL_1)</th>
<th>(SL_2)</th>
<th>(SL_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>(s,S)</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>26</td>
<td>27</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 6.9. \((s,S)\) policy, third order continuous delay, goal seeking behavior

Example 3: Consider parameters in Table 6.3. They satisfy \(Q/D = 4/3\) therefore period-4 oscillation must result as confirmed in Figure 6.10 and Figure 6.11.

Table 6.3. \((s,Q)\) parameters yielding P-4 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>(D)</th>
<th>(Q=(4/3)D)</th>
<th>(r_1)</th>
<th>(r_2)</th>
<th>(s)</th>
<th>(l_0)</th>
<th>(SL^1_0)</th>
<th>(SL^2_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>((s,Q))</td>
<td>2</td>
<td>4</td>
<td>16/3</td>
<td>2</td>
<td>4</td>
<td>28</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 6.10. \((s,Q)\) policy, P-4 oscillation phase map of \(I\) and \(SL^1\)
Figure 6.11. \((s,Q)\) policy, P-4 oscillation, phase map of \(I\) and \(SL^2\)

Example 4: Consider parameters in Table 6.4. They satisfy \(Q = 2 < D = 2.5\) therefore supply line stocks must exhibit goal seeking behavior while inventory must indefinitely decrease as confirmed in Figure 6.12.

Table 6.4. \((s,Q)\) parameters yielding goal seeking \(SL\) behavior, ever decreasing \(I\)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>D</th>
<th>Q</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>(\tau_3)</th>
<th>s</th>
<th>(l_0)</th>
<th>(SL^1)</th>
<th>(SL^2)</th>
<th>(SL^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>((s,Q))</td>
<td>3</td>
<td></td>
<td></td>
<td>2,5</td>
<td>2,5</td>
<td>3</td>
<td>20</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 6.12. \((s,Q)\) policy, third order, goal seeking behavior \(SL\), ever decreasing \(I\)
Example 5: Consider parameters in Table 6.5. They satisfy $Q = D = 2.5$ therefore goal seeking behavior must result as confirmed in Figure 6.13.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>$D$</th>
<th>$Q = D$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_3$</th>
<th>$S$</th>
<th>$I_0$</th>
<th>$SL_1^0$</th>
<th>$SL_2^0$</th>
<th>$SL_3^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>(s,Q)</td>
<td>3</td>
<td>2.5</td>
<td>2.5</td>
<td>1.5</td>
<td>2.5</td>
<td>3</td>
<td>20</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 6.13. $(s,Q)$ policy, third order continuous delay, goal seeking behavior

Example 6: Consider parameters in Table 6.6. They satisfy $R = 5$ therefore period-5 oscillation must result as confirmed in Figure 6.14 and Figure 6.15.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>$R$</th>
<th>$D$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$S$</th>
<th>$I_0$</th>
<th>$SL_1^0$</th>
<th>$SL_2^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>(R,S)</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>1.5</td>
<td>4</td>
<td>34</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 6.14. \((R, S)\) policy, P-5 oscillation phase map of \(I\) and \(SL^1\)

Figure 6.15. \((R, S)\) policy, P-5 oscillation phase map of \(I\) and \(SL^2\)

Example 7: Consider parameters in Table 6.7. They satisfy \(R = 1\) therefore goal seeking behavior must result as confirmed in Figure 6.16.

Table 6.7. \((R, S)\) parameters yielding goal seeking behavior

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>R</th>
<th>D</th>
<th>t₁</th>
<th>t₂</th>
<th>t₃</th>
<th>S</th>
<th>I₀</th>
<th>SL₀</th>
<th>SL₀^2</th>
<th>SL₀^3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>(R, S)</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>22</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 6.16. $(R, S)$ policy, third order continuous delay, goal seeking behavior

Example 8: Consider parameters in Table 6.8. They satisfy $R = 3$ and $(S - s)/(RD) = 7/6 \Rightarrow n = 2$ therefore period-6 oscillation must result as confirmed in Figure 6.17 and Figure 6.18.

Table 6.8. $(R, s, S)$ parameters yielding P-6 oscillation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>$R$</th>
<th>$D$</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$s$</th>
<th>$S$</th>
<th>$I_0$</th>
<th>$SL_0$</th>
<th>$SL_0'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>(R, s, S)</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>2</td>
<td>22</td>
<td>29</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 6.17. $(R, s, S)$ policy, P-6 oscillation phase map of $I$ and $SL'$
Figure 6.18. \((R, s, S)\) policy, P-6 oscillation phase map of \(I\) and \(SL^2\)

Example 9: Consider parameters in Table 6.9. They satisfy \(R = 1\) and \((\bar{S} - s) = 1 < D = 2\) therefore goal seeking behavior must result as confirmed in Figure 6.19.

Table 6.9. \((R, s, S)\) parameters yielding goal seeking behavior

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>(R)</th>
<th>(D)</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>(\tau_3)</th>
<th>(s)</th>
<th>(\bar{S})</th>
<th>(i_0)</th>
<th>(SL^1)</th>
<th>(SL^2)</th>
<th>(SL^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>((R, s, S))</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>32</td>
<td>33</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 6.19. \((R, s, S)\) policy, third order continuous delay, goal seeking behavior
6.5. Chapter Conclusion

The most important conclusion of this chapter comes from analysis of atomic structures, Section 6.1.1 and Section 6.1.2. In Section 6.1.1, it is shown that if the inflow of a first order continuous delay structure is constant (or it becomes constant after a transient period), both stock and outflow variables exhibit goal seeking behavior if outflow is proportional to stock. In Section 6.1.2, it is shown that if inflow of a first order continuous delay structure is periodic (or becomes periodic after a transient period), both stock and outflow variable display periodic behavior with the same period length as inflow if outflow is proportional to stock. To summarize, if outflow is periodic, stock and outflow variables exhibit same behavior as inflow. Note that these results are independent of the inventory policy and it is not even necessary for all delays to be continuous in the system.

These results are applied to inventory policies with $M^{th}$ order continuous delay structures. Supply line stocks are governed by the same equations as those atomics structures discussed in Sections 6.1.1 and 6.1.2. The first inflow to the system is order $O_k$ and from Section 4.5, it is known that for all inventory policies order shows either goal seeking or periodic behavior and since the system (for any policy) is completely composed of continuous delays all supply line stocks show same behavior as order; if order exhibits goal seeking behavior all supply line stocks exhibit goal seeking behavior, if order displays periodic behavior all supply line stocks display periodic behavior. Inventory stock has a different property than supply line stocks since its outflow is not proportional to inventory level. This situation is analyzed in detail in Section 6.2.3; inventory displays same behavior as order in all inventory policies except $(s,Q)$ when $Q < D$, in which case inventory indefinitely decreases while order exhibits goal seeking behavior. Results for each inventory policy are summarized in Section 6.3 for $M^{th}$ order continuous delay structure. Table 6.10 summarizes how supply line and inventory stocks behave in different conditions for each policy under $M^{th}$ order continuous delay structures.
Table 6.10. \( SL \) and \( I \) behavior under policies with \( M^{th} \) order continuous delays

<table>
<thead>
<tr>
<th>Policy</th>
<th>Condition</th>
<th>Behavior of supply line and inventory</th>
</tr>
</thead>
<tbody>
<tr>
<td>((s, S))</td>
<td>((S - s) \leq D)</td>
<td>Goal seeking</td>
</tr>
<tr>
<td></td>
<td>((S - s) &gt; D)</td>
<td>Period-( n ) oscillation where ( n ) is the smallest integer greater than or equal to the ratio ( (S - s)/D )</td>
</tr>
<tr>
<td></td>
<td>(Q &lt; D)</td>
<td>Inventory ever decreasing; supply line stocks goal seeking</td>
</tr>
<tr>
<td>((s, Q))</td>
<td>(Q = D)</td>
<td>Goal seeking</td>
</tr>
<tr>
<td></td>
<td>Rational (Q/D &gt; 1) (simplify (Q/D) until numerator and denominator are relatively prime)</td>
<td>Period-numerator oscillation</td>
</tr>
<tr>
<td></td>
<td>Irrational (Q/D &gt; 1) (approximate (Q/D) with a rational number whose numerator and denominator are relatively prime)</td>
<td>Approximately period-numerator oscillation</td>
</tr>
<tr>
<td>((R, S))</td>
<td>(R = 1)</td>
<td>Goal seeking</td>
</tr>
<tr>
<td></td>
<td>Integer (R &gt; 1)</td>
<td>Period-( R ) oscillation</td>
</tr>
<tr>
<td>((R, s, S))</td>
<td>(R = 1 ) and ((S - s) \leq D)</td>
<td>Goal seeking</td>
</tr>
<tr>
<td></td>
<td>Integer (R &gt; 1) and/or ((S - s) &gt; D)</td>
<td>Period-( nR ) oscillation where ( n ) is the smallest integer greater than or equal to the ratio ((S - s)/(RD))</td>
</tr>
</tbody>
</table>

Lastly, from Section 4.5, we know that orders behavior under \((s, S)\), \((R, S)\) and \((R, s, S)\) policies are very similar. Using this similarity with results of Sections 6.1.1 and 6.1.2, Table 6.11 summarizes supply line and effective inventory formulas for \((s, S)\), \((R, S)\) and \((R, s, S)\) policies where \( P \) is the period (in \((s, S)\) \( P = n \), in \((R, S)\) \( P = R \) and in \((R, s, S)\) \( P = nR \)). Table 5.33 is the special version of this table applied to first order continuous delay systems.
Table 6.11. \( SL \) and \( I \) formulas under \((s,S),(R,S)\) and \((R,s,S)\) policies

<table>
<thead>
<tr>
<th>Behavior</th>
<th>Supply line and Inventory Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goal seeking</td>
<td></td>
</tr>
<tr>
<td>( SL_e^1 = \tau_1 D ), ( SL_e^2 = \tau_2 D ), \ldots, ( SL_e^M = \tau_M D ), ( I_e = S - (\sum_{m=1}^{M} \tau_m) + 1)D )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Period-P oscillation</td>
<td></td>
</tr>
<tr>
<td>( SL_{k-1}^1 = \frac{PD}{1 - \alpha_1^p}, \quad SL_{k}^1 = \frac{\alpha_1 PD}{1 - \alpha_1^p}, \ldots, \quad SL_{k+p-2}^1 = \frac{\alpha_{p-1} PD}{1 - \alpha_1^p} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>( SL_{k-1}^2 = \frac{\sum_{j=1}^{p} (\alpha_2^{j-1} \cdot SL_{k-1-j}^1 / \tau_1)}{1 - \alpha_2^p}, \quad SL_{k}^2 = \frac{\sum_{j=1}^{p} (\alpha_2^{j-1} \cdot SL_{k-j}^1 / \tau_1)}{1 - \alpha_2^p}, \ldots, \quad SL_{k+p-2}^2 = \frac{\sum_{j=1}^{p} (\alpha_2^{j-1} \cdot SL_{k+p-2-j}^1 / \tau_1)}{1 - \alpha_2^p} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>( SL_{k-1}^M = \frac{\sum_{j=1}^{p} (\alpha_M^{j-1} \cdot SL_{k-1-j}^{M-1} / \tau_{M-1})}{1 - \alpha_M^p}, \ldots, \quad SL_{k+p-2}^M = \frac{\sum_{j=1}^{p} (\alpha_M^{j-1} \cdot SL_{k+p-2-j}^{M-1} / \tau_{M-1})}{1 - \alpha_M^p} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>( I_{k-1} = S - D - \sum_{m=1}^{M} SL_{k-1}^m, \quad I_k = S - 2D - \sum_{m=1}^{M} SL_{k}^m, \ldots, \quad I_{k+p-2} = S - PD - \sum_{m=1}^{M} SL_{k+p-2}^m )</td>
<td></td>
</tr>
</tbody>
</table>
7. INVENTORY DYNAMICS FORMULAS WITH DISCRETE (MIXED) DELAYS

In this chapter, inventory policies with discrete and mixed delay structures are analyzed. However when discrete delays are considered there are two general type of systems to be analyzed; continuous time discrete delay systems and discrete time discrete delay systems. For continuous time discrete delay systems, please refer to Appendix C where dynamic properties of continuous time discrete delay systems are discussed briefly, a stability result is given from literature and results are applied on an inventory model with continuous order policy. In this chapter, as in all other chapters of this thesis, we focus on discrete time models.

7.1. Simple Atomic Structures with Discrete Delay

The following discussion is independent of inflow structure. Stock flow diagram of simple discrete delay one stock atomic structure is shown in Figure 7.1.

![Figure 7.1. Discrete time, discrete delay one stock atomic structure](image)

This system is represented by the following set of equations.

\[ SL_k = SL_{k-1} + i_{k-1} - o_{k-1} \]
\[ o_k = i_{k-r} \]

At equilibrium \( SL_k - SL_{k-1} = i_{k-1} - o_{k-1} = i_{k-1} - i_{k-1-r} = 0 \). Therefore equilibrium of discrete delay stock can not be obtained using standard equilibrium analysis techniques.
Since \( o_j = i_{j-r} \) and for \( j \in [0, \tau - 1] \), \( i_{j-r} \) is undefined, we assume \( o_j = i_{j-r} = 0 \) for \( j \in [0, \tau - 1] \). From definition of a stock:

\[
SL_k = SL_0 + \sum_{j=0}^{k-1} (i_j - o_j) = SL_0 + \sum_{j=0}^{k-1} (i_j - i_{j-r})
\]

\[
= SL_0 + \sum_{j=0}^{k-1} i_j - \sum_{j=0}^{k-1} i_{j-r}
\]

\[
= SL_0 + \sum_{j=0}^{k-1} i_j - \sum_{j=0}^{k-\tau-1} i_j
\]

\[
= SL_0 + \sum_{j=k-\tau}^{k-1} i_j
\]

Therefore at any time point, level of discrete delay stock can be calculated from the following formulas.

\[
SL_k = \begin{cases} 
SL_0 + \sum_{j=0}^{k-1} i_j & \text{if } k < \tau \\
SL_0 + \sum_{j=k-\tau}^{k-1} i_j & \text{if } k \geq \tau 
\end{cases}
\] (7.1)

Discrete delay stocks forget all the past and remember just the recent past of \( \tau \) periods. Therefore we obtained the general rule which shows how discrete delay supply line stocks behave (this result is true for any inflow behavior).

Theoretically equilibrium point of any stock variable can be calculated from definition of stock, \( SL_e = SL_0 + \sum_{j=0}^{k_e-1} (i_j - o_j) \) where \( k_e \) is the time point when system reaches to the equilibrium. Though in most cases this technique cannot be applied due to special structure of discrete delay stocks, we can use this theoretical form. From Equation (7.1):
\[ SL_e = SL_0 + \sum_{j=k-\tau}^{k-1} i_j \]  \hspace{1cm} (7.2)

### 7.1.1. Constant Inflow Atomic Structure

If inflow is constant, that is \( i_j = i \), using Equation (7.1), at any time point discrete delay stock level can be calculated from:

\[ SL_k = \begin{cases} 
SL_0 + ik & \text{if } k < \tau \\
SL_0 + \tau & \text{if } k \geq \tau
\end{cases} \]

If \( k < \tau \), level of supply line increases and if \( k \geq \tau \), supply line level stabilizes at \( SL_0 + \tau \). Therefore supply line exhibits goal seeking behavior.

From Equation (7.2), when inflow is constant equilibrium of discrete delay stock is:

\[ SL_e = SL_0 + \sum_{j=k-\tau}^{k-1} i = SL_0 + \tau \]

As a result, if inflow eventually settles down to a constant, both discrete delay stock and outflow exhibit goal seeking behavior in the long term.

Example 1: Consider parameters \( i = 20, \ \tau = 5, \ SL_0 = 10 \). Both stock and outflow variables exhibit goal seeking behavior as confirmed in Figure 7.2.
Figure 7.2. Constant inflow, discrete time, discrete delay atomic structure

Example 2: Consider parameters $i_k = \begin{cases} \text{Uniform}(10, 20, 5) & \text{if } k < 15 \\ 5 & \text{else} \end{cases}$, $\tau = 5$ and $SL_0 = 0$. Although inflow is a random variable for some transient period, it eventually settles down to a constant value. Both stock and outflow variables exhibit goal seeking behavior as confirmed in Figure 7.3.

Figure 7.3. Transient variable inflow, discrete time, discrete delay atomic structure
7.1.2. Periodic Inflow Atomic Structure

One important property of periodic behaviors is that sum of \( R \) consecutive elements of a period-\( R \) series is constant wherever you start.

\[
\sum_{j=k}^{k+R-1} i_j = i_k + i_{k+1} + \cdots + i_{k+R-1} = A
\]  

(7.3)

If inflow is periodic, outflow certainly displays periodic behavior since it is just a delayed version of inflow. However, behavior of stock variable depends on the relationship between delay \( \tau \) and period length \( R \).

**Proposition:** If the delay \( \tau \) is an integer multiple of period \( R \) (i.e. \( \text{mod}(\tau, R) = 0 \) or \( \tau = nR \)) discrete delay stock exhibits goal seeking behavior with \( SL_e = SL_0 + nA \) where \( A \) is given by Equation (7.3) and \( n = \tau / R \). This can be shown in two different ways.

First Technique: From Equation (7.1):

\[
SL_k = SL_0 + \sum_{j=k-R}^{k-1} i_j = SL_0 + \sum_{j=k-nR}^{k-1} i_j = SL_0 + \sum_{j=k-nR}^{k-(n-1)R-1} i_j + \sum_{j=k-(n-2)R-1}^{k-(n-1)R-1} i_j + \sum_{j=k-R}^{A} i_j \quad , \quad k \geq nR
\]

\[
SL_k = SL_0 + nA \quad , \quad k \geq nR
\]

Second Technique: By definition \( o_k = i_{k-R} \). Since \( \tau = nR \), \( o_k = i_{k-nR} \). Since inflow displays periodic motion with \( R \), \( i_{k-nR} = i_k \).

\[
o_k = \begin{cases} 
i_k & \text{if } k \geq nR \\
0 & \text{else} \end{cases}
\]

(7.4)

We assume \( o_j = i_{j-nR} = 0 \) for \( j \in [0, nR - 1] \). Use Equation (7.4) with definition of a stock (accumulation of flows).
\[ SL_k = SL_0 + \sum_{j=0}^{k-1} (i_j - o_j) = SL_0 + \sum_{j=0}^{k-1} i_j - \sum_{j=0}^{k-1} o_j \]

\[ = SL_0 + \sum_{j=0}^{k-1} i_j - \sum_{j=nk}^{k-1} o_j \]

\[ = SL_0 + \sum_{j=0}^{k-1} i_j - \sum_{j=nk}^{k-1} i_j \]

\[ = SL_0 + \sum_{j=0}^{nk-1} i_j \]

\( SL_k \) where \( k \geq nR \), is a sum which is independent of \( k \) (\( k \) is not in the limits of summation). Again by partitioning this summation into \( R \) long portions, each equal to \( A \), same result is obtained as the first technique, \( SL_0 + nA \).

Whatever the inflow of discrete delay stock displays in the transient period if eventually it displays periodic motion the result found above is valid (\( j = 0 \) is not the starting point of simulation but the point where inflow of discrete delay stock starts to display periodic oscillation).

**Proposition:** If delay \( \tau \) is not an integer multiple of \( R \) (i.e. \( \text{mod}(\tau, R) \neq 0 \), including \( \tau < R \)) then discrete delay stock displays periodic oscillation.

Given that inflow displays period-\( R \) oscillation, how does stock variable given by Equation (7.1) behave if \( \text{mod}(\tau, R) \neq 0 \)? Equivalently, given a series making period-\( R \) oscillation, how does total of \( \tau \) consecutive points in that series behave? We try to answer this question from another way. It is known that inflow displays period-\( R \) oscillation. Since outflow is just a delayed version of inflow, outflow also displays period-\( R \) oscillation with the same numbers as inflow. How does difference of inflow and outflow (or netflow) behave?

Firstly, period of netflow can not be greater than \( R \) as shown below where \( nf \) represents netflow.
\[ i_k = i_{k+R} \quad \Rightarrow \quad n_{f_k} = n_{f_{k+R}} \]

Secondly, period of netflow cannot be one since \( \mod(\tau, R) \neq 0 \) (i.e. netflow cannot be constant). Netflow is a special difference series whose sum is zero during one period as shown below.

\[
\sum_{j=k}^{k+R-1} n_{f_j} = \sum_{j=k}^{k+R-1} (i_j - o_j) = \sum_{j=k}^{k+R-1} (i_j - i_{j+\tau}) \quad \text{for} \quad j \geq \tau
\]

\[
\sum_{j=k}^{k+R-1} i_j - \sum_{j=k}^{k+R-1} i_{j+\tau} = \sum_{j=k}^{k+R-1} i_j - \sum_{j=k-\tau}^{k+R-1} i_j = 0
\]

Now, assume netflow is constant.

\[ n_{f_j} = i_j - o_j = \mu \quad \text{for} \quad j = k, \ldots, k + R - 1 \]

where \( \mu \) is a real constant.

If we sum netflow during a period \( R \), we obtain the following.

\[
\sum_{j=k}^{k+R-1} n_{f_j} = \mu = \mu R
\]

Sum of netflow is shown to be zero during one period, therefore \( \mu R = 0 \Rightarrow \mu = 0 \Rightarrow n_{f_j} = i_j - o_j = 0 \) for \( j = k, \ldots, k + R - 1 \). This is only possible if \( \mod(\tau, R) = 0 \) which is a contradiction. As a result, if \( \mod(\tau, R) \neq 0 \), netflow cannot be constant.

At this point, it is known that if inflow displays period- \( R \) oscillation, period of netflow must be between \( (1, R] \).
Thirdly, period of netflow must be a divisor of $R$. It is shown that in any case netflow repeats itself in $R$ but it is not known whether $R$ is the smallest repetition cycle. Assume the smallest repetition cycle (i.e. the period) is $P$ where $P \in (1, R]$. Then \( \text{mod}(lR, P) = 0 \) for any positive integer $l$ must be satisfied. This means \( \text{mod}(R, P) = 0 \) is satisfied which is possible only if $P$ is a divisor of $R$.

Combining these three results, if inflow displays period-$R$ oscillation, netflow displays periodic-$P$ oscillation where $P \in (1, R]$ and $P$ is a divisor of $R$ (The important point is theoretically $P$ is not necessarily $R$, infact counter examples can be given where $P$ is not equal to $R$).

Although theoretically $P$ is not necessarily $R$, practically the probability of $P$ being not equal to $R$ is very small and it is unrealistic for a system to produce such a rare probability. This fact is demonstrated in simulation experiments (discrete delay stocks always display same periodic behavior as order).

Now, given that netflow displays period-$P$ oscillation where $P \in (1, R]$ and $P$ is a divisor of $R$, how does discrete delay stock behave? At time points $k$ and $k + P$, levels of stock variable can be calculated from definition of a stock as shown below.

\[
SL_k = SL_0 + \sum_{j=0}^{k-1} n f_j \quad \text{and} \quad SL_{k+P} = SL_0 + \sum_{j=0}^{k+P-1} n f_j
\]

Since sum of netflows during a period $R$ is zero and $P$ is a divisor of it, sum of netflows during period $P$ is also zero where $l = R/P$.

\[
\sum_{j=k}^{k+R-1} n f_j = 0 \Rightarrow l \cdot \sum_{j=k}^{k+P-1} n f_j = 0 \Rightarrow \sum_{j=k}^{k+P-1} n f_j = 0
\]

Using \( \sum_{j=k}^{k+P-1} n f_j = 0 \) and level of stock at time $k + P$:
\[ SL_{k+P} = SL_0 + \sum_{j=0}^{k+P-1} nf_j = SL_0 + \sum_{j=0}^{k-1} nf_j + \sum_{j=k}^{k+P-1} nf_j = SL_k \]

As a result, if inflow of discrete delay stock eventually displays period- \( R \) oscillation and \( \text{mod}(\tau, R) \neq 0 \), both stock and outflow display period-\( P \) oscillation where \( P \in (1, R) \) and \( P \) is a divisor of \( R \) (and most probably \( P = R \) in almost all practical situations).

To summarize, if inflow of discrete delay stock eventually displays period-\( R \) oscillation, outflow also displays period-\( R \) motion but behavior of stock variable depends on the relationship between delay \( \tau \) and period length \( R \).

\[
SL = \begin{cases} 
\text{Goal seeking with } SL_0 + nA \text{ where } A = \sum_{j=k}^{k+R-1} i_j \text{ and } n = \tau/R & \text{if } \text{mod}(\tau, R) = 0 \\
\text{Period - P oscillation} & \text{where } P \in (1, R) \text{ and } \text{mod}(R, P) = 0 \text{ if } \text{mod}(\tau, R) \neq 0
\end{cases}
\]

Example 1: Consider parameters \( i_k = \text{mod}(10,13,30,3), \ \tau = 4, \ SL_0 = 20 \). They satisfy \( \text{mod}(4,4) = 0 \) therefore stock variable must exhibit goal seeking behavior and outflow must display same periodic behavior as inflow as confirmed in Figure 7.4.

![Figure 7.4. Periodic inflow, discrete delay atomic structure, delay is 4](image-url)
Example 2: Consider parameters \( i_k = (3,15,10,25), \ \tau = 3, \ SL_0 = 15 \). They satisfy \( \text{mod}(3, 4) \neq 0 \) therefore both stock and outflow variables must show same periodic behavior as inflow as confirmed in

![Diagram](image)

Figure 7.5. Periodic inflow, discrete delay atomic structure, delay is 3

### 7.2. Inventory Dynamics of Mixed Delay Systems

Analysis of discrete time discrete delay atomic structures only applies to supply line stocks which represent the delay structures. Inventory stock does not represent a delay structure so the discussion made in Section 6.2.3 about inventory stock is still valid in discrete delay systems; inventory displays same behavior as order in all inventory policies except \((s, Q)\) when \(Q < D\), in which case order exhibits goal seeking behavior while inventory indefinitely decreases.

From Section 6.5, all continuous delay structures exhibit the same behavior as order.

Lastly if delay structure is discrete and order exhibits goal seeking behavior, supply line stocks also exhibit goal seeking behavior from Section 7.1.1. If delay structure is discrete and order displays periodic behavior, supply line behavior is defined by the relationship between delay \(\tau\) and period length \(R\); if \(\text{mod}(\tau, R) = 0\) supply line stocks exhibit goal seeking behavior and if \(\text{mod}(\tau, R) \neq 0\) they display periodic behavior as order.
In Section 7.1, we obtained the formula how to calculate level of a supply line stock for any inflow behavior. However in our inventory policies in any case inflow is either goal seeking or periodic (if we consider first supply line stock inflow is the order \( O_k \), for a general supply line stock \( m \) inflow is outflow of supply line stock \( m-1 \)). If we combine this result with those in Chapter 6 we can find level of all stock variables for \((s,S)\), \((R,S)\) and \((R,s,S)\) policies under higher order mixed delay systems. The following algorithm is obtained by combining results of Sections 6.1.1, 6.1.2 and 7.1. This algorithm is a generalized version of Table 6.11 which summarizes supply line and inventory value formulas for \((s,S)\), \((R,S)\) and \((R,s,S)\) policies under higher order continuous delay systems.

```plaintext
prevcon=0;
for(int m=1;m<M; m++)
{
    if (typeof(m)===continuous)
    {
        for(i=k-1;i<k+P-2;i++)
        {
            \[ SL_i^m = \frac{\sum_{j=1}^{P} \alpha_m^{j-1} SL_j^{prevcon} \sum_{i=prevcon+1}^{\infty} \tau_i}{1-\alpha_m^P} \]

            prevcon=m;
        }
    }
    else //if (typeof(m)===discrete)
    {
        for(i=k-1;i<k+P-2;i++)
        {
            \[ SL_i^m = SL_i^0 + \sum_{j=1}^{i} (SL_j^{prevcon} \sum_{i=prevcon+1}^{\infty} \tau_i) \]
        }
    }

```
where $SL^0$ is the order $O_k$, $\tau_o = 1$ and variable prevcon stores the number of last continuous supply line stock while algorithm advances.

From Chapter 4, effective inventory behavior is known. Therefore after obtaining supply line stock levels from the algorithm above, by subtracting them from effective inventory values, inventory level is obtained. We apply this algorithm to Example 11 and compare analytical results with simulation results.

### 7.3. Simulation Experiments

Example 1: Delay is discrete. Consider parameters in Table 7.1. They satisfy $(S - s)/D = 7/2$ therefore order and inventory must display P-4 oscillation and since mod(2, 4) $\neq 0$ supply line (discrete) must display P-4 as confirmed in Figure 7.6.

**Table 7.1. (s, S) parameters with first order discrete delay**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of Delay</th>
<th>D</th>
<th>Q</th>
<th>$\tau$</th>
<th>s</th>
<th>S</th>
<th>$I_0$</th>
<th>$SL^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>(s, S)</td>
<td>1</td>
<td>2</td>
<td>7</td>
<td>2</td>
<td>6</td>
<td>13</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

![Figure 7.6. (s, S) policy, first order discrete delay, P-4 oscillation](image)
Example 2: First delay is discrete while second delay is continuous. Consider parameters in Table 7.2. They satisfy \((S - s)/D = 8/3\) therefore order, inventory and second supply line (continuous) display period-3 oscillation and since \(\text{mod}(6, 3) = 0\), first supply line stock (discrete) exhibits goal seeking behavior as confirmed in Figure 7.7.

Table 7.2. \((s, S)\) parameters with second order mixed delay

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of Delay</th>
<th>D</th>
<th>Q</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>(s)</th>
<th>S</th>
<th>I_0</th>
<th>SL_1^0</th>
<th>SL_2^0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>(s, S)</td>
<td>2</td>
<td>3</td>
<td>8</td>
<td>6</td>
<td>1,2</td>
<td>24,6</td>
<td>32,6</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

![Figure 7.7. \((s, S)\) policy, second order mixed delay](image)

Example 3: First delay is continuous while second and third delays are discrete. Consider parameters in Table 7.3. They satisfy \((S - s) = 1 < D = 2.5\) therefore goal seeking behavior must result as confirmed in Figure 7.8.

Table 7.3. \((s, S)\) parameters with third order mixed delay

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of Delay</th>
<th>D</th>
<th>Q</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>(\tau_3)</th>
<th>s</th>
<th>S</th>
<th>I_0</th>
<th>SL_1^0</th>
<th>SL_2^0</th>
<th>SL_3^0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>(s, S)</td>
<td>3</td>
<td>2.5</td>
<td>1</td>
<td>1.5</td>
<td>3</td>
<td>6</td>
<td>28,75</td>
<td>29,75</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 7.8. \((s, S)\) policy, third order mixed delay

Example 4: Both delays are discrete. Consider parameters in Table 7.4. They satisfy \(Q = D = 6\) therefore goal seeking behavior must result as confirmed in Figure 7.9.

Table 7.4. \((s, Q)\) parameters with second order discrete delay

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>(D)</th>
<th>(Q = D)</th>
<th>(r_1)</th>
<th>(r_2)</th>
<th>(s)</th>
<th>(I_0)</th>
<th>(SL_0^1)</th>
<th>(SL_0^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>((s, Q))</td>
<td>2</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>42</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 7.9. \((s, Q)\) policy, second order discrete delay
Example 5: First supply line is continuous while second and third ones are discrete. Consider parameters in Table 7.5. They satisfy $Q/D = 3$ therefore order, inventory and first supply line (continuous) display period-3 oscillation and since mod(6,3) = 0 and mod(3,3) = 0 second (discrete) and third supply line (discrete) stocks exhibit goal seeking behavior as confirmed in Figure 7.10.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>D</th>
<th>$Q=3D$</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$\tau_3$</th>
<th>$s$</th>
<th>$l_0$</th>
<th>$SL^1_0$</th>
<th>$SL^2_0$</th>
<th>$SL^3_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>$(s,Q)$</td>
<td>3</td>
<td></td>
<td>1,5</td>
<td>4,5</td>
<td>3</td>
<td>6</td>
<td>3</td>
<td>19,5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

![Graph](image)

Figure 7.10. $(s,Q)$ policy, third order mixed delay

Example 6: All delays are discrete. Consider parameters in Table 7.6. They satisfy $Q = 2 < D = 5$ therefore order and supply line stocks (discrete) exhibit goal seeking behavior while inventory indefinitely decreases as confirmed in Figure 7.11.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>D</th>
<th>$Q$</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$\tau_3$</th>
<th>$s$</th>
<th>$l_0$</th>
<th>$SL^1_0$</th>
<th>$SL^2_0$</th>
<th>$SL^3_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>$(s,Q)$</td>
<td>3</td>
<td></td>
<td>5</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>50</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 7.11. \((s,Q)\) policy, third order discrete delay

Example 7: Both delays are discrete. Consider parameters in Table 7.7. They satisfy \(R = 1\) therefore goal seeking behavior must result as confirmed in Figure 7.12.

Table 7.7. \((R,S)\) parameters with second order discrete delay

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>R</th>
<th>D</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>S</th>
<th>(I_0)</th>
<th>(SL^1)</th>
<th>(SL^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>(R,S)</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>28</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 7.12. \((R,S)\) policy, second order discrete delay
Example 8: First delay is continuous while second delay is discrete. Consider parameters in Table 7.8. They satisfy $R = 3/2$ therefore order, inventory and first supply line (continuous) display period-3 oscillation and since $\text{mod}(2, 3) \neq 0$, second supply line (discrete) displays period-3 oscillation as confirmed in Figure 7.13. However, as stated, to choose a noninteger $R$ is not meaningful when discrete time step, $k$, is specified as a small enough step.

<table>
<thead>
<tr>
<th>Table 7.8. $(R,S)$ parameters with second order mixed delay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
</tr>
<tr>
<td>Values</td>
</tr>
</tbody>
</table>

![Figure 7.13. $(R,S)$ policy, second order mixed delay](image)

Example 9: First and second delays are discrete while third one is continuous. Consider parameters in Table 7.9. They satisfy $R = 4$ therefore order, inventory and third supply line (continuous) display period-4 oscillation and since $\text{mod}(8, 4) = 0$ and $\text{mod}(4, 4) = 0$ first (discrete) and second supply line (discrete) stocks exhibit goal seeking behavior as confirmed in Figure 7.14 and Figure 7.15.
Table 7.9. \((R,S)\) parameters with third order mixed delay

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>R</th>
<th>D</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>(\tau_3)</th>
<th>(S)</th>
<th>(I_0)</th>
<th>(SL_1^0)</th>
<th>(SL_2^0)</th>
<th>(SL_3^0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R,S)</td>
<td>3</td>
<td>4</td>
<td>18</td>
<td>8</td>
<td>1.5</td>
<td>4</td>
<td>1.5</td>
<td>18,50</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

![Graph 1](image1)

**Figure 7.14. \((R,S)\) policy, third order mixed delay**

![Graph 2](image2)

**Figure 7.15. \((R,S)\) policy, third order mixed delay, phase map of \(I\) and \(SL^3\)**
Example 10: First delay is continuous while second delay is discrete. Consider parameters in Table 7.10. They satisfy $R = 1$ and $(S - s) = 1 < D = 2$ goal seeking behavior must result as confirmed in Figure 7.16.

Table 7.10. $(R, s, S)$ parameters with second order mixed delay

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>$R$</th>
<th>$D$</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$s$</th>
<th>$S$</th>
<th>$I_0$</th>
<th>$SL^1_0$</th>
<th>$SL^2_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>$(R, s, S)$</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>2</td>
<td>18</td>
<td>19</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 7.16. $(R, s, S)$ policy, second order mixed delay

Example 11: First and third delays are discrete while second one is continuous. Consider parameters in Table 7.11. They satisfy $R = 3$ and $(S - s)/(RD) = 15/12$ therefore order, inventory and second supply line (continuous) display period-6 oscillation and since mod$(12, 6) = 0$ and mod$(6, 6) = 0$ both first (discrete) and third supply lines (discrete) exhibit goal seeking behavior as confirmed in Figure 7.17 and Figure 7.18.

Table 7.11. $(R, s, S)$ parameters with third order mixed delay

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model Type</th>
<th>Order of delay</th>
<th>$R$</th>
<th>$D$</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$\tau_3$</th>
<th>$s$</th>
<th>$S$</th>
<th>$I_0$</th>
<th>$SL^1_0$</th>
<th>$SL^2_0$</th>
<th>$SL^3_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>$(R, s, S)$</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>12</td>
<td>3</td>
<td>6</td>
<td>100</td>
<td>115</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 7.17. $(R,s,S)$ policy, third order mixed delay

Figure 7.18. $(R,s,S)$ policy, third order mixed delay, phase map of $I$ and $SL^2$

Now we will apply the algorithm and compare results with simulation results: $P=6$ (oscillation length), $M=3$ (order of supply line), $prevcon=0$, and $Order=(24,0,0,0,0,0)$.

First step: $m=1$, first supply line is discrete and $SL^1_0 = 0$. 
\[ SL_{k-1} = \sum_{j=k-13}^{k-2} \frac{SL^0_{k-1-j}}{\tau_0} = \sum_{j=k-13}^{k-2} \frac{O_{k-1-j}}{\tau_0} = 48 \]

\[ SL_k = \sum_{j=k-12}^{k-1} \frac{SL^0_{k-j}}{\tau_0} = \sum_{j=k-12}^{k-1} \frac{O_{k-j}}{\tau_0} = 48 \]

\[ SL_{k+1} = \sum_{j=k-11}^{k} \frac{SL^0_{k+1-j}}{\tau_0} = \sum_{j=k-11}^{k} \frac{O_{k+1-j}}{\tau_0} = 48 \]

\[ SL_{k+2} = \sum_{j=k-10}^{k+1} \frac{SL^0_{k+2-j}}{\tau_0} = \sum_{j=k-10}^{k+1} \frac{O_{k+2-j}}{\tau_0} = 48 \]

\[ SL_{k+3} = \sum_{j=k-9}^{k+2} \frac{SL^0_{k+3-j}}{\tau_0} = \sum_{j=k-9}^{k+2} \frac{O_{k+3-j}}{\tau_0} = 48 \]

\[ SL_{k+4} = \sum_{j=k-8}^{k+3} \frac{SL^0_{k+4-j}}{\tau_0} = \sum_{j=k-8}^{k+3} \frac{O_{k+4-j}}{\tau_0} = 48 \]

Second step: prevcon=0, m=2 and second supply line is continuous.

\[ SL^2_{k-1} = \sum_{j=1}^{6} \frac{\alpha^2_{j-1} SL^0_{k-1-j}}{\tau_0} = \sum_{j=1}^{6} \frac{\frac{2}{3} \alpha^2_{j-1} O_{k-1-j}}{\tau_0} = 26.31 \]

\[ SL^2_k = \sum_{j=1}^{6} \frac{\alpha^2_{j-1} SL^0_{k-j}}{\tau_0} = \sum_{j=1}^{6} \frac{\frac{2}{3} \alpha^2_{j-1} O_{k-j}}{\tau_0} = 17.54 \]

\[ SL^2_{k+1} = \sum_{j=1}^{6} \frac{\alpha^2_{j-1} SL^0_{k+1-j}}{\tau_0} = \sum_{j=1}^{6} \frac{\frac{2}{3} \alpha^2_{j-1} O_{k+1-j}}{\tau_0} = 11.693 \]

\[ SL^2_{k+2} = \sum_{j=1}^{6} \frac{\alpha^2_{j-1} SL^0_{k+2-j}}{\tau_0} = \sum_{j=1}^{6} \frac{\frac{2}{3} \alpha^2_{j-1} O_{k+2-j}}{\tau_0} = 7.9955 \]

\[ SL^2_{k+3} = \sum_{j=1}^{6} \frac{\alpha^2_{j-1} SL^0_{k+3-j}}{\tau_0} = \sum_{j=1}^{6} \frac{\frac{2}{3} \alpha^2_{j-1} O_{k+3-j}}{\tau_0} = 5.197 \]

\[ SL^2_{k+4} = \sum_{j=1}^{6} \frac{\alpha^2_{j-1} SL^0_{k+4-j}}{\tau_0} = \sum_{j=1}^{6} \frac{\frac{2}{3} \alpha^2_{j-1} O_{k+4-j}}{\tau_0} = 3.465 \]

prevcon=2

Third step: prevcon=2, m=3, third supply line is discrete and \( SL^3_0 = 0 \)
\[
SL_{k-1}^2 = \sum_{j=k-7}^{k-2} SL_{k-1,j}^2 / \tau_2 = 24 \\
SL_k^2 = \sum_{j=k-6}^{k-1} SL_{k,j}^2 / \tau_2 = 24 \\
SL_{k+1}^2 = \sum_{j=k-5}^{k+1} SL_{k+1,j}^2 / \tau_2 = 24 \\
SL_{k+2}^2 = \sum_{j=k-4}^{k+1} SL_{k+2,j}^2 / \tau_2 = 24 \\
SL_{k+3}^2 = \sum_{j=k-3}^{k+2} SL_{k+3,j}^2 / \tau_2 = 24 \\
SL_{k+4}^2 = \sum_{j=k-2}^{k+3} SL_{k+4,j}^2 / \tau_2 = 24
\]

From Chapter 4, effective inventory values are known. If we simply subtract supply line values from effective inventory we obtain inventory levels. These analytical results are confirmed by the simulation results shown in Table 7.12.

\[
I_{k-1} = S - D - \sum_{m=1}^{3} SL_{k-1,m} = 12.69 \\
I_k = S - 2D - \sum_{m=1}^{3} SL_{k,m} = 17.46 \\
I_{k+1} = S - 3D - \sum_{m=1}^{3} SL_{k+1,m} = 19.308 \\
I_{k+2} = S - 4D - \sum_{m=1}^{3} SL_{k+2,m} = 19.2045 \\
I_{k+3} = S - 5D - \sum_{m=1}^{3} SL_{k+3,m} = 17.803 \\
I_{k+4} = S - 6D - \sum_{m=1}^{3} SL_{k+4,m} = 15.535
\]

Table 7.12. Simulation results third order mixed delay

<table>
<thead>
<tr>
<th>Time</th>
<th>229</th>
<th>230</th>
<th>231</th>
<th>232</th>
<th>233</th>
<th>234</th>
</tr>
</thead>
<tbody>
<tr>
<td>First supply line</td>
<td>48</td>
<td>48</td>
<td>48</td>
<td>48</td>
<td>48</td>
<td>48</td>
</tr>
<tr>
<td>Second supply line</td>
<td>26,31</td>
<td>17,54</td>
<td>11,69</td>
<td>7,8</td>
<td>5,2</td>
<td>3,46</td>
</tr>
<tr>
<td>Third supply line</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>Inventory</td>
<td>12,69</td>
<td>17,46</td>
<td>19,31</td>
<td>19,2</td>
<td>17,8</td>
<td>15,54</td>
</tr>
</tbody>
</table>
7.4. Chapter Conclusion

In Section 7.1.1, it is shown that if the inflow of a discrete delay structure eventually exhibits constant behavior, both stock and outflow variables exhibit goal seeking behavior. In Section 7.1.2, it is shown that if inflow of a discrete delay structure eventually displays periodic behavior, outflow variable exhibits same behavior as inflow since it is just a delayed version of inflow. However, behavior of stock variable is determined by the relationship between delay \( \tau \) and period length \( R \); if \( \text{mod}(\tau, R) \neq 0 \), the stock displays oscillatory behavior and if \( \text{mod}(\tau, R) = 0 \), it exhibits goal seeking behavior. Note that these results are independent of the inventory policy and it is not even necessary for all delays to be discrete in the system. Therefore results of Chapters 6 and 7 can be applied together in mixed delay systems. From Section 6.5, all continuous supply lines (delay structures) exhibit same behavior as order. From Section 4.5, behavior of order is known for each policy. If order exhibits goal seeking behavior, all delay structures, continuous or discrete, exhibit goal seeking behavior. If order displays periodic behavior, all continuous supply lines display periodic oscillation while discrete supply lines display periodic behavior if \( \text{mod}(\tau, R) \neq 0 \) and exhibit goal seeking behavior if \( \text{mod}(\tau, R) = 0 \).

Inventory stock does not represent a delay structure so the discussion made in Section 6.2.3 about inventory stock is entirely valid here; inventory displays same behavior as order in all inventory policies except \( (s, Q) \) when \( Q < D \), in which case while order exhibits goal seeking behavior, inventory decreases.

Order behaviors are summarized in Table 8.2 and behavior of stock variables (both supply line and inventory stocks) with respect to order behavior are summarized in Table 8.1. Using Table 8.1 and Table 8.2 together, for any policy, any delay type and any order we can state behavior of all stock variables.

We can also find level of any stock variable in higher order mixed delay systems using the algorithm in Section 7.2.
8. CONCLUSION AND FUTURE WORK

Inventory control is important in successful management because companies cannot afford to have money tied up in excess inventories. Four policies frequently used in inventory control management are Order Point - Order Up to Level \((s, S)\) policy, Order Point - Order Quantity \((s, Q)\) policy, Review Period - Order Up to Level \((R, S)\) policy and \((R, s, S)\) policy. In this thesis, dynamics that result from the application of these standard inventory management policies are analyzed under different delay structures: first order continuous delay, \(M^{th}\) order continuous delay and discrete (or mixed) delay structures.

This thesis fills a gap between two different groups; system dynamics and optimization circles. While System Dynamics literature has its own ordering rules (such as anchor and adjust) and is not generally interested in these optimal inventory policies, optimization circles are not interested in dynamics of these policies. Therefore, in this thesis we combine these two different views.

In the first phase of this thesis, behaviors of effective inventory (inventory position) and orders are derived for each policy as they are the building blocks of the remaining analysis. In the second phase, inventory dynamics formulas are derived under the assumption of first order continuous delay structure. In the third phase, behavior types (such as goal seeking, periodic) of supply line and inventory stocks of each policy are derived under the assumption of \(M^{th}\) order continuous delay structure. In the fourth phase, discrete delay structure is analyzed and behavior types of supply line and inventory stocks are derived for higher order mixed delay structures.

Effective inventory and orders dynamics for each inventory policy are analyzed in Chapter 4. Behavior types of effective inventory and orders for each policy under different conditions are summarized in Table 4.15.

Dynamics of inventory policies under first order continuous delay structures are analyzed in Chapter 5. Behavior types of supply line and inventory of \((s, Q)\) policy are
summarized in Table 5.30. Supply line and inventory value formulas for \((s,S),(R,S)\) and \((R,s,S)\) are summarized in Table 5.29, Table 5.31 and Table 5.32 respectively.

Dynamics of inventory policies under \(M^{th}\) order continuous delay structures are analyzed in Chapter 6. Behavior types of supply line and inventory stocks for each policy with \(M^{th}\) order continuous delay structure are summarized in Table 6.10. Supply line and inventory value formulas are summarized in Table 6.11 for \((s,S),(R,S)\) and \((R,s,S)\) policies. Table 6.11 is a generalized version of Table 5.33 for higher order continuous delay inventory systems.

Dynamics of inventory policies under discrete (mixed) delay structures are analyzed in Chapter 7. It is shown that if the order exhibits goal seeking behavior, all discrete supply lines exhibit goal seeking behavior. If the order displays periodic behavior, discrete supply line stocks display periodic behavior if the delay \(\tau\) is not an integer multiple of order period \(R\) (i.e. \(\text{mod}(\tau,R) \neq 0\)) and exhibit goal seeking behavior if delay \(\tau\) is an integer multiple of order period \(R\) (i.e. \(\text{mod}(\tau,R) = 0\)).

Table 8.1 and Table 8.2 summarize combined results of Chapter 6 and Chapter 7. Using these two tables, we are able to determine behavior of the supply line stocks (either as continuous delays or discrete delays) and inventory stock for any policy, under any given condition.

We can also find level of any supply line stock variable for \((s,S)\), \((R,S)\) and \((R,s,S)\) policies under higher order mixed delay systems (the most general case) using the algorithm discussed in Section 7.2 which is obtained by combining results of Sections 6.1.1, 6.1.2 and 7.1. After finding supply line values from the algorithm stated above, inventory values can be obtained by subtracting the supply line from effective inventory values.

\[
I_{k-1} = S - D - \sum_{m=1}^{M} SL^m_{k-1}, \quad I_k = S - 2D - \sum_{m=1}^{M} SL^m_k, \ldots, I_{k+p-2} = S - PD - \sum_{m=1}^{M} SL^m_{k+p-2}
\]
As a future work, inventory dynamics of retailer under different forms of demand patterns may be investigated; more generally, for a given order patterns in a supply chain to analyze behavior of the preceding agent stock. A second research direction may be to analyze time continuous inventory systems with discrete delay structures.

Table 8.1. Summary of behavior types of orders

<table>
<thead>
<tr>
<th>Policy</th>
<th>Condition</th>
<th>Behavior of orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(s, S)$</td>
<td>$(S - s) \leq D$</td>
<td>Goal seeking</td>
</tr>
<tr>
<td></td>
<td>$(S - s) &gt; D$</td>
<td>Period-$n$ oscillation where $n$ is the smallest integer greater than or equal to the ratio $(S - s)/D$</td>
</tr>
<tr>
<td>$(s, Q)$</td>
<td>$Q &lt; D$</td>
<td>Goal seeking</td>
</tr>
<tr>
<td></td>
<td>$Q = D$</td>
<td>Goal seeking</td>
</tr>
<tr>
<td></td>
<td>Rational $Q/D &gt; 1$ (simplify $Q/D$ until numerator and denominator are relatively prime)</td>
<td>Period-numerator oscillation</td>
</tr>
<tr>
<td></td>
<td>Irrational $Q/D &gt; 1$ (approximate by a rational number whose numerator and denominator are relatively prime)</td>
<td>Approximately period-numerator oscillation</td>
</tr>
<tr>
<td>$(R, S)$</td>
<td>$R = 1$</td>
<td>Goal seeking</td>
</tr>
<tr>
<td></td>
<td>Integer $R &gt; 1$</td>
<td>Period-$R$ oscillation</td>
</tr>
<tr>
<td>$(R, s, S)$</td>
<td>$R = 1$ and $(S - s) \leq D$</td>
<td>Goal seeking</td>
</tr>
<tr>
<td></td>
<td>Integer $R &gt; 1$ and/or $(S - s) &gt; D$</td>
<td>Period-$nR$ oscillation where $n$ is the smallest integer greater than or equal to the ratio $(S - s)/(RD)$</td>
</tr>
</tbody>
</table>

Table 8.2. Behavior of different stock types with respect to order behavior

<table>
<thead>
<tr>
<th>Order behavior</th>
<th>Continuous supply line (delay)</th>
<th>Discrete supply line (delay)</th>
<th>Inventory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goal seeking</td>
<td>Goal seeking</td>
<td>Goal seeking</td>
<td>Goal seeking. Only in $(s, Q)$, ever decreasing if $Q &lt; D$</td>
</tr>
<tr>
<td>Period-P oscillation</td>
<td>Period-P oscillation</td>
<td>Periodic if $\text{mod}(r, P) \neq 0$</td>
<td>Goal seeking if $\text{mod}(r, P) = 0$</td>
</tr>
</tbody>
</table>
APPENDIX A: EFFECTIVE INVENTORY DYNAMICS UNDER
\((R,s,S)\) POLICY

**Proposition:** \(E_I\) can not indefinitely remain greater than order point \(s\). From Equation (4.20) if \(E_I > s\) and/or \(\text{mod}(k,R) \neq 0\) then order is zero and \(D>0\). Therefore, even if effective inventory (inventory position) is greater than order point \(s\), it decreases until a time point which is a multiple of \(R\) where effective inventory is less than or equal to order point \(s\); i.e. at some time point \(R_k\), \(E_{I_k} \leq s\) is certainly satisfied.

**Proposition:** If \(R=1\) and \((S-D) \leq s\), \(E_I\) exhibits goal seeking behavior and stays constant at \(S-D\). \((R,s,S)\) with \(R=1\) is exactly equivalent to \((s,S)\) since first condition in if statement in Equation (4.20) is always satisfied and \(E_I\) under \((s,S)\) exhibits goal seeking behavior and stays constant at \(S-D\) if \((S-s) \leq D\) from Section 4.1.1. If effective inventory stays constant at \(S-D\), the order \(O_k\) stays constant at \(D\) from Equation (4.20).

**Proposition:** Effective inventory can not indefinitely remain on one side of order point \(s\) when \((S-s) > D\) (or \((S-D) > s\)). At some time point \(R_k\), \(E_{I_k} \leq s\) is certainly satisfied. From Equation (4.21) if \(E_{I_k} \leq s \Rightarrow E_{I_{k+1}} = S-D > s\).

**Proposition:** If \((S-s) > D\) and/or \(R>1\), effective inventory displays period-\(nR\) oscillation where \(n\) is the smallest integer greater than or equal to ratio \((S-s)/(RD)\) and it drops to order point \(s\) or lower exactly one time which is also a multiple of \(R\) in any period.

In \((R,s,S)\) policy, order decision is taken at time points which are multiples of \(R\) and where effective inventory is at order point \(s\) or lower. Therefore if time is not a multiple of \(R\), even if we do not know the comparison of effective inventory with order point \(s\), effective inventory drops by \(D\) amount. As an example, using Equations (4.21) and (4.22):
\[ EI_{Rk} \leq s \Rightarrow EI_{Rk+1} = S - D \]
\[
\begin{align*}
EI_{Rk+1} \leq s & \Rightarrow EI_{Rk+2} = EI_{Rk+1} - D = S - 2D \\
EI_{Rk+2} \leq s & \Rightarrow EI_{Rk+3} = EI_{Rk+2} - D = S - 3D \\
& \vdots \\
EI_{Rk+n} \leq s & \Rightarrow EI_{Rk+n+1} = EI_{Rk+n} - D = S - RD
\end{align*}
\]

Now, assume the following behavior sequence for effective inventory.

\[ EI_{Rk} \leq s \]
\[ EI_{Rj} > s, j = k+1, \ldots, k+n-1 \]
\[ EI_{R(k+n)} \leq s \]
\[ EI_{Rj} > s, j = k+n+1, k+n+2, \ldots, k+n+l-1 \]
\[ EI_{R(k+n+l)} \leq s \] \hspace{1cm} (A.1)

There is no loss of generality in the proposed behavior sequence. There is no restriction on \( n \) and \( l \). Most importantly it is not assumed that this behavior sequence repeats itself or complete (it is just a portion of effective inventory behavior). Using Equations (4.21), (4.22) and (A.1):

\[ EI_{Rk} \leq s \Rightarrow EI_{Rk+n} = S - RD \]
\[
\begin{align*}
EI_{Rk+n} > s & \Rightarrow EI_{Rk+n+2} = S - 2RD \\
EI_{Rk+n+2} > s & \Rightarrow EI_{Rk+n+3} = S - 3RD \\
& \vdots \\
EI_{Rk+(n-2)} > s & \Rightarrow EI_{Rk+(n-1)} = S - (n-1)RD \\
EI_{Rk+(n-1)} > s & \Rightarrow EI_{Rk+n} = S - nRD \\
EI_{Rk+n} \leq s & \Rightarrow EI_{Rk+n+1} = S - RD \\
EI_{Rk+n+1} > s & \Rightarrow EI_{Rk+n+2} = S - 2RD \\
& \vdots \\
EI_{Rk+n+(l-2)} > s & \Rightarrow EI_{Rk+n+(l-1)} = S - (l-1)RD \\
EI_{Rk+n+(l-1)} > s & \Rightarrow EI_{Rk+n+l} = S - lRD \\
EI_{Rk+n+l} \leq s & \Rightarrow S - lRD \leq s
\end{align*}
\] \hspace{1cm} (A.2)

There are two cases to be analyzed; \( n = 1 \) and \( n > 1 \).
If \( n = 1 \), the middle steps where effective inventory is greater than order point \( s \) at multiples of \( R \) are eliminated, that is Equations (A.1) and (A.2) turn into Equation (A.3).

\[
\begin{align*}
EI_{Rk} &\leq s \Rightarrow EI_{Rk+R} = S - RD \\
EI_{Rk+R} &\leq s \Rightarrow EI_{Rk+R+R} = S - RD \\
\{ &\quad \text{or} \quad \} \\
EI_{Rk+R+R+2R} &> s \Rightarrow EI_{Rk+R+(l-1)R} = S - (l-1)RD \\
\vdots \\
EI_{Rk+R+(l-2)R} &> s \Rightarrow EI_{Rk+R+(l-1)R} = S - (l-1)RD \\
EI_{Rk+R+(l-1)R} &> s \Rightarrow EI_{Rk+R+lR} = S - lRD \\
EI_{Rk+R+lR} &\leq s \Rightarrow S - lRD \leq S
\end{align*}
\]

\[\Rightarrow\]

\[
\begin{align*}
EI_{Rk+R} &\leq s \Rightarrow S - RD \leq s \\
\{ &\quad \text{or} \quad \} \\
EI_{Rk+R+R} &> s \Rightarrow S - RD > s \\
\vdots \\
EI_{Rk+R+(l-1)R} &> s \Rightarrow S - (l-1)RD > s \\
EI_{Rk+R+lR} &\leq s \Rightarrow S - lRD \leq S
\end{align*}
\]

It can be easily shown that the following 2 equations can not happen simultaneously unless \( l = n = 1 \).

\[
\begin{align*}
EI_{Rk+R} &= S - RD \leq s \\
EI_{Rk+R+(l-1)R} &= S - (l-1)RD > s
\end{align*}
\]

(A.4)

If \( l \neq n \), there is only one possibility: \( l > 1 \). \( 1 < l \Rightarrow 1 \leq l-1 \Rightarrow S - RD \geq S - (l-1)RD \). Equation (A.4) clearly contradicts this inequality. Therefore \( l \) must be equal to 1, equal to \( n \); i.e. \( n = l = 1 \).

By definition, period is the length from (including) a time point which is a multiple of \( R \) where \( EI \) is less than or equal to \( s \) and the next time point which is also a multiple of \( R \) and where \( EI \) is less than or equal to \( s \). In Equation (A.3), at time \( Rk \) \( EI_{Rk} \leq s \) and at time \( Rk + R \) \( EI_{Rk+R} \leq s \), therefore period length is \( R \). However, this is not a contradiction with our proposition since \( n \) is 1, period-\( nR \) becomes period-\( R \).

If \( n > 1 \), it can be easily shown that the following 4 equations can not happen simultaneously unless \( l = n \).

\[
\begin{align*}
EI_{Rk+(n-1)R} &= S - (n-1)RD > s \\
EI_{Rk+nR} &= S - nRD \leq s \\
\{ &\quad \text{or} \quad \} \\
EI_{Rk+nR+(l-1)R} &= S - (l-1)RD > s \\
EI_{Rk+nR+lR} &= S - lRD \leq s
\end{align*}
\]

(A.5)
Assume \( n \neq l \) and \( n \neq 1 \), there are 2 possibilities.

Either \( l < n \Rightarrow l \leq n - 1 \Rightarrow S - lRD \geq S - (n - 1)RD \). Equation (A.6) clearly contradicts this inequality.

\[
EI_{Rk+(n-1)R} = S - (n-1)RD > s \quad \text{and} \quad EI_{Rk+nR+lR} = S - lRD \leq s \quad \text{(A.6)}
\]

Or \( n < l \Rightarrow n \leq l - 1 \Rightarrow S - nRD \geq S - (l - 1)RD \). Equation (A.7) clearly contradicts this inequality.

\[
EI_{Rk+nR} = S - nRD \leq s \quad \text{and} \quad EI_{Rk+nR+(l-1)R} = S - (l - 1)RD > s \quad \text{(A.7)}
\]

As a result it is shown that if \( l \neq n \), Equation (A.5) is a self-contradictory set, therefore \( l \) must be equal to \( n \). \( l = n \) means number of time points that are multiples of \( R \) where effective inventory is greater than order point \( s \) between two time points which are multiples of \( R \) and where effective inventory is less than or equal to order point \( s \) must be constant. So, even if we continue to write the behavior sequence (remember it is not asserted that it is complete), we must obey the upper result. That is, effective inventory displays periodic oscillation in which at only one time point multiple of \( R \), it drops to order point \( s \) or lower.

Using definition of periodicity stated above, in Equation (A.1) at time \( Rk \) \( EI_{Rk} \leq s \) and at time \( R(k+n) \) \( EI_{R(k+n)} \leq n \), therefore period length is \( nR \) where \( n \) is given by the following equation.

\[
EI_{R(k+n)} = S - nRD \leq s \Rightarrow \frac{S - s}{RD} \leq n
\]

Thus, period of \( EI \) is \( nR \) where \( n \) is the smallest number greater than or equal to ratio \( (S - s)/(RD) \).
If effective inventory displays period-$nR$ oscillation so does the order $O_k$ from Equation (4.20). Order decision is made once (at this point the order quantity is $S - EI = S - (S - nRD) = nRD$) in a period and at other points in that period the order $O_k$ is zero. Thus, order displays period-$nR$ oscillation with points $(nRD, 0, ..., 0)$ where number of zeros is $nR - 1$.

If $R$ is not an integer, the discussion made in Section 4.3.1 is also valid under $(R, s, S)$ policy since that discussion does not depend on the inventory policy considered. However, as stated, to choose a noninteger $R$ is not meaningful when discrete time step, $k$, is specified as a small enough step.
Appendix B: Dynamics of Inventory Under \((R, s, S)\) Policy

\(R\) is assumed to be a positive integer greater than 1. From difference calculus it is known that \(nR\) simultaneous order-\(nR\) difference equations produce period-\(nR\) oscillation if absolute value of each eigen value of each of the \(nR\) equations is less than or equal to 1. Therefore first, equation pairs starting from \((Rk + 2)\) to \((Rk + 2nR)\) are written. Then inventory and supply line at point \((Rk + nR + 1)\) in terms of inventory and supply line at \((Rk + 1)\), inventory and supply line at point \((Rk + nR + 2)\) in terms of inventory and supply line at \((Rk + 2)\), ..., inventory and supply line at point \((Rk + 2nR)\) in terms of inventory and supply line at \((Rk + R)\) are obtained.

We define \(n\) as the smallest integer greater than ratio \((S - s)/(RD)\) so that \(S - nRD \leq s\).

For all points other than \((Rk + nR + 1)\), equation pairs are:

\[
\begin{align*}
I_{j+1} &= I_j + SL_j / \tau - D \\
SL_{j+1} &= SL_j - SL_j / \tau = \alpha SL_j \\
\Rightarrow EI_{j+1} &= EI_j - D \quad j = \{Rk + 1, \ldots, Rk + 2nR - 1\} - \{Rk + nR\}
\end{align*}
\]

(B.1)

At time \((Rk + nR + 1)\), equation pair changes since \(\text{mod}(Rk + nR, R) = 0\) and effective inventory (inventory position) is less than or equal to \(s\) from definition of \(n\) \((O_{Rk+nR} = S - EI_{Rk+nR})\).

\[
\begin{align*}
I_{Rk+nR+1} &= I_{Rk+nR} + \frac{SL_{Rk+nR}}{\tau} - D \\
SL_{Rk+nR+1} &= SL_{Rk+nR} + S - I_{Rk+nR} - SL_{Rk+nR} - \frac{SL_{Rk+nR}}{\tau} \Rightarrow EI_{Rk+nR+1} = S - D \\
= S - I_{Rk+nR} - \frac{SL_{Rk+nR}}{\tau}
\end{align*}
\]

(B.2)
From Equations (B.1) and (B.2), for time points greater than \((Rk + nR + 1)\) effective inventory values are:

\[
EI_{Rk+nR+1} = S - D \\
EI_{Rk+nR+2} = EI_{Rk+nR+1} - D = S - 2D \\
EI_{Rk+nR+3} = EI_{Rk+nR+2} - D = S - 3D \\
\vdots \\
EI_{Rk+nR+nR} = EI_{Rk+nR+nR-1} - D = S - nRD
\]  
(B.3)

Note that at time point \((Rk + nR + 1)\) equations are changing. Inventory and supply line at a general \((Rk + nR + 1)\) in terms of inventory and supply line at \((Rk + l)\) are derived then by putting numbers instead of \(l\) others are obtained. We start with inventory, using Equations (B.1) and (B.2):

\[
I_{Rk+nR+l} = I_{Rk+l} + \sum_{j=Rk+l}^{Rk+nR+l-1} \frac{SL_j}{\tau} - nRD = I_{Rk+l} + \frac{\sum_{j=Rk+l}^{Rk+nR+l-1} SL_j}{\tau} - nRD
\]

\[
I_{Rk+nR+l} = I_{Rk+l} + \frac{(1 + \alpha + \ldots + \alpha^{nR-l})SL_{Rk+l} + (1 + \alpha + \ldots + \alpha^{l-2})SL_{Rk+nR+l}}{\tau} - nRD
\]

\[
I_{Rk+nR+l} = I_{Rk+l} + \frac{1 - \alpha^{nR-l+1}}{\tau} - nRD
\]

\[
I_{Rk+nR+l} = I_{Rk+l} + (1 - \alpha)SL_{Rk+l} + (1 - \alpha^l)SL_{Rk+nR+l} - nRD
\]

\[
I_{Rk+nR+l} = I_{Rk+l} + (1 - \alpha^{nR-l+1})SL_{Rk+l} + (1 - \alpha^l)(S - I_{Rk+nR} - \frac{SL_{Rk+nR}}{\tau}) - nRD
\]

\[
I_{Rk+nR+l} = I_{Rk+l} - (1 - \alpha^l)I_{Rk+nR} + (1 - \alpha^{nR-l+1})SL_{Rk+l} - (1 - \alpha^l)(1 - \alpha)\alpha^{nR-l}SL_{Rk+l} + (1 - \alpha^l)S - nRD
\]

\[
I_{Rk+nR+l} = I_{Rk+l} - (1 - \alpha^l)I_{Rk+nR} + (1 - \alpha^{nR-l+1} + \alpha^{nR-l} - \alpha^l)SL_{Rk+l} + (1 - \alpha^l)S - nRD
\]  
(B.4)

\(I_{Rk+nR}\) must be written in terms of \(I_{Rk+l}\) and \(SL_{Rk+l}\).
$$I^*_{Rk+nR} = I_{Rk+l} + \sum_{j \in Rk+l} \frac{SL_j}{\tau} - (nR-l)D = I_{Rk+l} + \frac{(1+\alpha + \ldots + \alpha^{nR-l-1})SL_{Rk+l}}{\tau} - (nR-l)D$$

$$I^*_{Rk+nR} = I_{Rk+l} + \frac{1-\alpha^{nR-l}}{\tau} SL_{Rk+l} - (nR-l)D$$

$$I^*_{Rk+nR} = I_{Rk+l} + (1-\alpha^{nR-l})SL_{Rk+l} - (nR-l)D$$

(B.5)

Plug Equation (B.5) into Equation (B.4).

$$I^*_{Rk+nR+1} = I_{Rk+l} - (1-\alpha^{l-1})I^*_{Rk+nR} + (1-\alpha^{nR-l} + \alpha^{nR-l} - \alpha^{nR})SL_{Rk+l} + (1-\alpha^{l-1})S - nRD$$

$$I^*_{Rk+nR+1} = I_{Rk+l} - (1-\alpha^{l-1})(I_{Rk+l} + (1-\alpha^{nR-l})SL_{Rk+l} - (nR-l)D) + (1-\alpha^{nR-l} + \alpha^{nR-l} - \alpha^{nR})SL_{Rk+l} + (1-\alpha^{l-1})S - nRD$$

$$I^*_{Rk+nR+l} = \alpha^{l-1}I_{Rk+l} + (1-\alpha^{nR-l} + \alpha^{nR-l} - \alpha^{nR} - (1-\alpha^{l-1})(1-\alpha^{nR-l}))(1-\alpha^{l-1})(1-\alpha^{nR-l}))SL_{Rk+l} + (1-\alpha^{l-1})S - nRD$$

$$I^*_{Rk+nR+l} = \alpha^{l-1}I_{Rk+l} + (\alpha^{l-1} - \alpha^{nR})SL_{Rk+l} + S - lD - \alpha^{l-1}(S + (nR-l)D)$$

(B.6)

Now we obtain supply line, using Equation (B.1) and (B.2):

$$\begin{align*}
SL_{Rk+nR+l+1} &= \alpha SL_{Rk+nR+l-1} \\
SL_{Rk+nR+l-1} &= \alpha SL_{Rk+nR+l-2} \\
&\vdots \\
SL_{Rk+nR+2} &= \alpha SL_{Rk+nR+1} \\
\Rightarrow SL_{Rk+nR+l} &= \alpha^{l-1}SL_{Rk+nR+l} \\
\end{align*}$$

(B.7)

$$SL_{Rk+nR+1} = S - I_{Rk+nR} - \frac{SL_{Rk+nR}}{\tau}$$

(B.8)

$$\begin{align*}
SL_{Rk+nR} &= \alpha SL_{Rk+nR-1} \\
SL_{Rk+nR-1} &= \alpha SL_{Rk+nR-2} \\
&\vdots \\
SL_{Rk+l+1} &= \alpha SL_{Rk+l} \\
\Rightarrow SL_{Rk+nR} &= \alpha^{nR-1}SL_{Rk+l} \\
\end{align*}$$

(B.9)

Now, consider Equations (B.7), (B.8) and (B.9) together:
\[ SL_{Rk+nR+l} = \alpha^{l-1}SL_{Rn+nR+l} = \alpha^{l-1}(S - I_{Rk+nR} - \frac{SL_{Rk+nR}}{\tau}) = -\alpha^{l-1}I_{Rk+nR} - \alpha^{l-1}(1 - \alpha)SL_{Rk+nR} + \alpha^{l-1}S \]

(B.10)

Plug Equation (B.5) into Equation (B.10).

\[ SL_{Rk+nR+l} = -\alpha^{l-1}(I_{Rk+l} + (1 - \alpha^{nR-1})SL_{Rk+l} - (nR - l)D) - \alpha^{l-1}(1 - \alpha)\alpha^{nR-1}SL_{Rk+l} + \alpha^{l-1}S \]

\[ SL_{Rk+nR+l} = -\alpha^{l-1}I_{Rk+l} + (\alpha^{l-1} + \alpha^{nR-1} - (1 - \alpha)\alpha^{nR-1})SL_{Rk+l} + \alpha^{l-1}S + \alpha^{l-1}(nR - l)D \]

\[ SL_{Rk+nR+l} = -\alpha^{l-1}I_{Rk+l} - (\alpha^{l-1} - \alpha^{nR})SL_{Rk+l} + \alpha^{l-1}(S + (nR - l)D) \]

(B.11)

Both \( I_{Rk+nR+l} \) and \( SL_{Rk+nR+l} \) are written in terms of \( I_{Rk+l} \) and \( SL_{Rk+l} \), Equations (B.4) and (B.10). A quick check is \( I_{Rk+nR+l} + SL_{Rk+nR+l} = S - lD \) which is consistent with Equation (B.3).

\[
\begin{pmatrix}
I_{Rk+nR+l} \\
SL_{Rk+nR+l}
\end{pmatrix}
= \begin{pmatrix}
\alpha^{l-1} & (\alpha^{l-1} - \alpha^{nR}) \\
-\alpha^{l-1} & -(\alpha^{l-1} - \alpha^{nR})
\end{pmatrix}
\begin{pmatrix}
I_{Rk+l} \\
SL_{Rk+l}
\end{pmatrix}
+ \begin{pmatrix}
S - lD - \alpha^{l-1}(S + (nR - l)D) \\
\alpha^{l-1}(S + (nR - l)D)
\end{pmatrix}
\]

From Equation (5.63):

\[
\begin{cases}
\text{either } \lambda = 0 \\
\text{or } \lambda = \alpha^{l-1} - (\alpha^{l-1} - \alpha^{nR}) = \alpha^{nR}
\end{cases}
\]

For any value of \( l \), \( \lambda = 0 \) or \( \lambda = \alpha^{nR} \) and since \( |\alpha| = |1 - 1/\tau| \leq 1 \), roots of the characteristic equation are in the unit circle, so the system is stable (therefore absolute value of eigen value of each of \( nR \) equations is less than or equal to 1). We are interested in the long term behavior of the system, so we do not try to find the homogenous solution since it vanishes as time increases. (Remember homogenous solution consists of power of roots of characteristic equation which are shown to be smaller than 1). Since nonhomogenous part is composed of constant terms, try a constant vector for inventory and supply line values at \( (Rk + nR + l) \) in the long term.

\[
k = \begin{pmatrix}
I_{Rk+nR+l} \\
SL_{Rk+nR+l}
\end{pmatrix}
\]
From Equation (5.64):

\[
k = \frac{1}{1 + (\alpha^{l-1} - \alpha^{nR}) - \alpha^{l-1}} \left( \frac{(\alpha^{l-1} - \alpha^{nR})}{1 - \alpha^{l-1}} \right) \left( S - lD - \alpha^{l-1} (S + (nR - l)D) \right)
\]

\[
SL_{Rk+nR+l} = \frac{n\alpha^{l-1}}{1 - \alpha^{nR}} RD
\]  

(B.12)

If we add Equations (B.4) and (B.10), \( I_{Rk+nR+l} + SL_{Rk+nR+l} = S - lD \). Subtract Equation (B.12).

\[
I_{Rk+nR+l} = S - lD - \frac{n\alpha^{l-1}}{1 - \alpha^{nR}} RD
\]  

(B.13)

So both inventory and supply line values at \((Rk + nR + l)\) are found. Now, plug \(l = 1, 2, \ldots, nR\) into Equations (B.6), (B.11), (B.12) and (B.13).

\[
\begin{align*}
I_{Rk+nR+l} &= I_{Rk+l} + (1 - \alpha^{nR})SL_{Rk+l} - nRD \\
SL_{Rk+nR+l} &= -I_{Rk+l} - (1 - \alpha^{nR})SL_{Rk+l} + S + (nR - l) \\
&\Rightarrow \left\{ \begin{array}{l}
\text{either } \lambda = 0 \\
\text{or } \lambda = \alpha^{nR} \Rightarrow SL_{Rk+nR+l} = \frac{n}{1 - \alpha^{nR}} RD
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
I_{Rk+nR+l+2} &= \alpha I_{Rk+l+2} + (\alpha - \alpha^{nR})SL_{Rk+l+2} + S - 2D - \alpha(S + (nR - 2)D) \\
SL_{Rk+nR+l+2} &= -\alpha I_{Rk+l+2} - (\alpha - \alpha^{nR})SL_{Rk+l+2} + \alpha(S + (nR - 2)D) \\
&\Rightarrow \left\{ \begin{array}{l}
\text{either } \lambda = 0 \\
\text{or } \lambda = \alpha^{nR} \Rightarrow SL_{Rk+nR+l+2} = \frac{n\alpha^{l-1}}{1 - \alpha^{nR}} RD
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
I_{Rk+nR+l} &= \alpha^{l-1} I_{Rk+l} + (\alpha^{l-1} - \alpha^{nR})SL_{Rk+l} + S - lD - \alpha^{l-1} (S + (nR - l)D) \\
SL_{Rk+nR+l} &= -\alpha^{l-1} I_{Rk+l} - (\alpha^{l-1} - \alpha^{nR})SL_{Rk+l} + \alpha^{l-1}(S + (nR - l)D) \\
&\Rightarrow \left\{ \begin{array}{l}
\text{either } \lambda = 0 \\
\text{or } \lambda = \alpha^{nR} \Rightarrow SL_{Rk+nR+l} = \frac{n\alpha^{l-1}}{1 - \alpha^{nR}} RD
\end{array} \right.
\end{align*}
\]

\[
\vdots
\]

\[
\vdots
\]
\[
\begin{align*}
I_{Rk+nR+nR} &= \alpha_{nR} - I_{Rk+nR} + (\alpha_{nR} - \alpha_{nR})SL_{Rk+nR} + S - nRD - \alpha_{nR}S \\
SL_{Rk+nR+nR} &= -\alpha_{nR} - I_{Rk+nR} - (\alpha_{nR} - \alpha_{nR})SL_{Rk+nR} + \alpha_{nR}S \\
\implies \begin{cases} 
\text{either } \lambda = 0 \\
\text{or } \lambda = \alpha_{nR} \Rightarrow SL_{Rk+nR+nR} = \frac{n\alpha_{nR} - 1}{1 - \alpha_{nR} RD}
\end{cases}
\end{align*}
\]

Thus, supply line values are found from \((Rk + nR + 1)\) to \((Rk + 2nR)\). Effective inventory values are also known from Equation (B.3). Inventory values can be obtained easily by subtracting supply line values from effective inventory figures. Both supply line and inventory equations are given in Equation (B.14).

\[
\begin{align*}
SL_{Rk+nR+1} &= \frac{n}{1 - \alpha_{nR} RD} \\
SL_{Rk+nR+2} &= \frac{n\alpha}{1 - \alpha_{nR} RD} \\
SL_{Rk+nR+3} &= \frac{n\alpha^2}{1 - \alpha_{nR} RD} \\
\vdots \\
SL_{Rk+nR+nR} &= \frac{n\alpha_{nR} - 1}{1 - \alpha_{nR} RD} \\
I_{Rk+nR+1} &= S - D - \frac{n}{1 - \alpha_{nR} RD} \\
I_{Rk+nR+2} &= S - 2D - \frac{n\alpha}{1 - \alpha_{nR} RD} \\
I_{Rk+nR+3} &= S - 3D - \frac{n\alpha^2}{1 - \alpha_{nR} RD} \\
\vdots \\
I_{Rk+nR+nR} &= S - nRD - \frac{n\alpha_{nR} - 1}{1 - \alpha_{nR} RD}
\end{align*}
\]

(B.14)
APPENDIX C: CONTINUOUS TIME DISCRETE DELAY SYSTEMS

In this chapter, we briefly present dynamical properties of simple continuous time, discrete delay systems. A stability result is given from the literature and results are applied on an inventory model with continuous order policy.

The characteristic feature of a system with time lags is the dynamics at a certain time does not only depend on the instantaneous state of the system but also on some past values. The dependence on the past can have different forms [6]. If the process under investigation depends on the full history of the system over a certain time interval, mathematical formulation leads to distributed delay differential equations. If the process under investigation depends on a finite number of past values of the state variable, mathematical formulation leads to discrete delay differential equations. The latter is a special case of the former. This latter type is also called delay differential equations or difference-differential equations [9].

According to the problem analyzed the unknown variable, \( y \), can be a function of a single independent variable where derivatives appear as ordinary or more than one independent variable where derivatives appear as partial. Assuming unknown variable is a function of a single independent variable, time, according to the complexity of the phenomenon, delays (or lags) may be just constants, constant delay case, or functions of time, \( \tau_i = \tau_i(t) \), the variable or time dependent delay case, or even functions of time and \( y \), \( \tau_i = \tau_i(t, y(t)) \), the state dependent delay case [9].

C.1. Equilibrium and Stability Analysis

Equilibrium analysis of delay differential equations (continuous time discrete delay systems) is same as ordinary differential equations after plugging zero instead of all delays. However we may not be able to get the equilibrium points using standard equilibrium analysis techniques.
In the stability analysis of delay differential equations there are two possible approaches that have been considered in the literature. One approach consists in finding conditions such that the problem is stable for all or for some classes of delays, typically for all constant delays. The second approach consists in finding weaker conditions such that the desired stability property is guaranteed for the specific given (in general constant) delay. The two concepts of stability are actually different and the former is referred as “stability for all delays” or “delay independent stability” and to the latter as “stability for fixed delay” or “delay dependent stability” [9].

Here, we give only the stability result of constant coefficient constant delay (pure delay) differential equations. For more information about stability analysis in delay differential equations refer to [9].

\[
y'(t) = \mu y(t - \tau), \quad t \geq t_0 \\
y(t) = \phi(t), \quad t \leq t_0
\]  \hspace{1cm} (C.1)

where \( y \) is the unknown variable, \( y'(t) \) is the derivative of unknown variable \( y \) at time \( t \) defined as \( \frac{dy}{dt} \), \( \phi \) is the initial function, \( \mu \) is a constant, \( \tau \) is delay and \( t_0 \) is the initial time.

For pure delay differential equations, we can get only delay dependent conditions. Linear scalar equation with constant delay \( \tau \) in Equation (C.1) is asymptotically stable for any initial function \( \phi(t) \) if Equation (C.2) is satisfied [9].

\[
0 < -\tau \mu < \pi / 2
\]  \hspace{1cm} (C.2)

C.2. Simple Atomic Structures with Discrete Delay

Since the analyzed atomic structures do not have feedback structures, results are almost same as Section 7.1.
The following discussion is independent of inflow structure. Stock flow diagram of simple discrete delay one stock atomic structure is shown in Figure C.1.

![Stock Flow Diagram](image)

**Figure C.1.** Continuous time, discrete delay one stock atomic structure

This system is represented by the following set of equations.

\[
\begin{align*}
SL'(t) &= i(t) - o(t) \\
o(t) &= i(t - \tau)
\end{align*}
\]  
(C.3)

where \( SL(t), i(t), o(t) \) represent supply line, inflow rate and outflow rate at time \( t \) respectively and \( \tau \) is delay.

From Section C.1, in order to find equilibrium points we insert zero for delay \( \tau \) in Equation (C.3), \( SL'(t) = i(t) - o(t) = 0 \Rightarrow i(t) - i(t) = 0 \). Therefore equilibrium can not be obtained using standard equilibrium analysis techniques.

Since \( o(t) = i(t - \tau) \) and for \( t \in [0, \tau) \), \( i(t - \tau) \) is undefined, we assume \( o(t) = i(t - \tau) = 0 \) for \( t \in [0, \tau) \). From definition of a stock:

\[
\begin{align*}
SL(t) &= SL(0) + \int_0^t (i(t) - o(t)) dt = SL(0) + \int_0^t (i(t) - i(t - \tau)) dt \\
&= SL(0) + \int_0^t i(t) dt - \int_0^t i(t - \tau) dt \\
&= SL(0) + \int_0^\tau i(t) dt - \int_0^{t-\tau} i(t) dt \\
&= SL(0) + \int_{t-\tau}^t i(t) dt
\end{align*}
\]
Therefore at any time point, level of discrete delay stock can be calculated from the following formulas.

\[
SL(t) = \begin{cases} 
SL(0) + \int_0^t i(t)dt & \text{if } t < \tau \\
SL(0) + \int_{\tau - \tau}^\tau i(t)dt & \text{if } t \geq \tau
\end{cases} 
\] (C.4)

Discrete delay stocks forget all the past and remember just the recent past of \( \tau \) periods.

Theoretically equilibrium point of any stock variable can be calculated from definition of a stock, \( SL_e = SL(0) + \int_0^{t_e} (i(j) - o(j))dj \) where \( t_e \) is the time point when system reaches to equilibrium. Though in most cases such a technique can not be applied, due to special structure of discrete delay stocks, we can use this theoretical form. From Equation (C.4):

\[
SL_e = SL(0) + \int_{\tau - \tau}^{t_e} i(j)dj 
\] (C.5)

C.2.1. Constant Inflow Atomic Structure

If inflow is constant, that is \( i(t) = i \), using Equation (C.4), at any time point discrete delay stock level can be calculated from:

\[
SL(t) = \begin{cases} 
SL(0) + it & \text{if } t < \tau \\
SL(0) + i\tau & \text{if } t \geq \tau
\end{cases}
\]

If \( t < \tau \), level of supply line increases and if \( t \geq \tau \), supply line level stabilizes at \( SL(0) + i\tau \). Therefore supply line exhibits goal seeking behavior.
From Equation (C.5), when inflow is constant equilibrium of discrete delay stock is:

\[
SL_e = SL(0) + \int_{-\tau}^{t} idt = SL(0) + i\tau
\]  \hspace{1cm} (C.6)

As a result, if inflow eventually settles down to a constant, both discrete delay stock and outflow exhibit goal seeking behavior in the long term.

Example 1: Consider parameters \( i = 20, \tau = 5, SL(0) = 0 \). They satisfy constant inflow therefore both stock and outflow variables exhibit goal seeking behavior as confirmed in Figure C.2.

![Figure C.2. Constant inflow, continuous time, discrete delay system](image)

Example 2: Consider parameters \( i(t) = \begin{cases} \text{Uniform}(10,20,5) & \text{if} \,(t < 15) \\ 5 & \text{else} \end{cases} \), \( \tau = 5 \) and \( SL(0) = 0 \). They satisfy constant inflow case (although inflow is a random variable for some transient period, it eventually settles down to a constant value), both stock and outflow variables exhibit goal seeking behavior as confirmed in Figure C.3.
C.2.2. Periodic Inflow Atomic Structure

In this case inflow is periodic. One important property of periodic behaviors is:

\[ \int_{j}^{j+R} i(t) \, dt = A \]  \hspace{1cm} (C.7)

If inflow is periodic, outflow certainly displays periodic behavior since it is just a delayed version of inflow. However, behavior of stock depends on the relationship between delay \( \tau \) and period length \( R \).

**Proposition:** If the delay \( \tau \) is an integer multiple of period \( R \) (i.e. \( \text{mod}(\tau, R) = 0 \) or \( \tau = nR \)) discrete delay stock exhibits goal seeking behavior with \( SL_e = SL(0) + nA \) where \( A \) is given by Equation (C.7) and \( n = \tau / R \). This can be shown in two different ways.

First Technique: Using Equation (C.4):
\[ SL(t) = SL(0) + \int_{t-nR}^{t} i(j) \, dj = SL(0) + \int_{t-(n-1)R}^{t} i(j) \, dj \]
\[ SL(t) = SL(0) + \int_{t-nR}^{t-(n-1)R} i(j) \, dj + \int_{t-(n-1)R}^{t} i(j) \, dj + \cdots + \int_{t-R}^{t} i(j) \, dj, \quad t \geq nR \]
\[ SL(t) = SL(0) + nA, \quad t \geq nR \]

Second Technique: By definition \( o(t) = i(t-\tau) \). Since \( \tau = nR \), \( o(t) = i(t-nR) \).

Since inflow displays periodic motion with \( R \), \( i(t-nR) = i(t) \).

\[ o(t) = \begin{cases} i(t) & \text{if } t \geq nR \\ 0 & \text{else} \end{cases} \quad (C.8) \]

We assume \( o(t) = i(t-\tau) = 0 \) for \( t \in [0,nR) \). Use Equation (C.8) with the definition of a stock (accumulation of flows).

\[ SL(t) = SL(0) + \int_{0}^{t} (i(t) - o(t)) \, dt = SL(0) + \int_{0}^{t} i(t) \, dt - \int_{0}^{t} o(t) \, dt \]
\[ = SL(0) + \int_{0}^{t} i(t) \, dt - \int_{nR}^{t} o(t) \, dt \]
\[ = SL(0) + \int_{0}^{t} i(t) \, dt - \int_{nR}^{t} i(t) \, dt \]
\[ = SL(0) + \int_{0}^{nR} i(t) \, dt \]

\( SL(t) \) where \( t \geq nR \), is a sum which is independent of \( t \) (\( t \) is not in the limits of summation). Again by partitioning this summation into \( R \) long portions, each equal to \( A \), we get the same result as first technique, \( SL(0) + nA \).

Whatever the inflow of discrete delay stock displays in the transient period if eventually it displays periodic motion the result found above is valid (\( j = 0 \) is not the starting point of simulation but the point where the inflow of discrete delay stock starts to display periodic oscillation).
Proposition: If delay $\tau$ is not an integer multiple of $R$ (i.e. $\text{mod}(\tau, R) \neq 0$, including $\tau < R$) then discrete delay stock displays periodic oscillation.

Given that inflow displays period-$R$ oscillation, how does stock variable given by Equation (C.4) behave if $\text{mod}(\tau, R) \neq 0$? Equivalently, given a series making period-$R$ oscillation, how does total of $\tau$ consecutive points in that series behave? We try to answer this question from another way. It is known that inflow displays period-$R$ oscillation. Since outflow is just a delayed version of inflow, outflow also displays period-$R$ oscillation with the same numbers as inflow. How does difference of inflow and outflow (or netflow) behave?

Firstly, period of netflow can not be greater than $R$ as shown below where $nf$ represents netflow.

\[
i(t) = i(t + R) \\
o(t) = o(t + R) \Rightarrow nf(t) = nf(t + R)
\]

Secondly, period of netflow can not be one since $\text{mod}(\tau, R) \neq 0$ (i.e. netflow can not be constant). Netflow is a special difference series whose sum is zero during one period as shown below.

\[
\int_j^{j+R} nf(t)dt = \int_j^{j+R} (i(i) - o(i))dt = \int_j^{j+R} (i(t) - i(t - \tau))dt \text{ for } t \geq \tau \\
\int_j^{j+R} i(t)dt - \int_j^{j+R} i(t - \tau)dt = \int_j^{j+R} i(t)dt - \int_{j-\tau}^{j+R-\tau} i(t)dt = 0
\]

Now, assume netflow is constant.

\[
nf(j) = i(j) - o(j) = \mu \text{ for } j \in [t, t + R]
\]

where $\mu$ is a real constant.
If we sum netflow during a period $R$, we obtain the following.

$$\int_{j}^{j+R} nf(t)dt = \int_{j}^{j+R} \mu dt = \mu R$$

Sum of netflow is shown to be zero during one period, therefore $\mu R = 0 \Rightarrow \mu = 0 \Rightarrow nf(j) = i(j) - o(j) = 0$ for $j \in [t, t + R]$. This is only possible if mod($\tau, R$) = 0 which is a contradiction. As a result, if mod($\tau, R$) $\neq 0$, netflow can not be constant.

At this point, it is known that if inflow displays period-$R$ oscillation, period of netflow must be between $(l, R]$.

Thirdly, period of netflow must be a divisor of $R$. It is shown that in any case netflow repeats itself in $R$ but it is not known whether $R$ is the smallest repetition cycle. Assume the smallest repetition cycle (i.e. the period) is $P$ where $P \in (l, R]$. Then mod($lR, P$) = 0 for any positive integer $l$ must be satisfied. This means mod($R, P$) = 0 is satisfied which is possible only if $P$ is a divisor of $R$.

Combining these three results, if inflow displays period-$R$ oscillation, netflow displays periodic-P oscillation where $P \in (l, R]$ and $P$ is a divisor of $R$ (The important point is theoretically $P$ is not necessarily $R$, infact counter examples can be given where $P$ is not equal to $R$).

Although theoretically $P$ is not necessarily $R$, practically the probability of $P$ being not equal to $R$ is very small and it is unrealistic for a system to produce such a rare probability.

Now, given that netflow displays period-$P$ oscillation where $P \in (l, R]$ and $P$ is a divisor of $R$, how does discrete delay stock behave? At time points $t$ and $t + P$, levels of stock variable can be calculated from definition of a stock as shown below.
\[
SL(t) = SL(0) + \int_0^t nf(t)dt \quad \text{and} \quad SL(t + P) = SL(0) + \int_0^{t+P} nf(t)dt
\]

Since sum of netflows during a period \( R \) is zero and \( P \) is a divisor of it, sum of netflows during period \( P \) is also zero.

\[
\int_t^{t+R} nf(t)dt = 0 \Rightarrow l \cdot \int_t^{t+P} nf(t)dt = 0 \Rightarrow \int_t^{t+P} nf(t)dt = 0
\]

where \( l = R/P \).

Using \( \int_t^{t+P} nf(t)dt = 0 \) and level of stock at time \( t + P \):

\[
SL(t + P) = SL(0) + \int_0^{t+P} nf(t)dt = 0 = SL(0) + \int_0^t nf(t)dt + \int_t^{t+P} nf(t)dt = 0 \Rightarrow SL(t + P) = SL(t)
\]

As a result, if inflow of discrete delay stock eventually displays period-\( R \) oscillation and \( \text{mod}(\tau, R) \neq 0 \), both stock and outflow display period-\( P \) oscillation where \( P \in (1, R] \) and \( P \) is a divisor of \( R \) (and most probably \( P = R \) in almost all practical situations).

To summarize, if inflow of discrete delay stock eventually displays period-\( R \) oscillation, outflow also displays period-\( R \) motion but behavior of stock variable depends on the relationship between delay and period length.

\[
SL_c = \begin{cases} 
\text{Goal seeking with } SL(0) + nA \quad \text{where } A = \int_0^{j+R} i(t)dt \quad \text{and } n = \tau/R \quad \text{if } \text{mod}(\tau, R) = 0 \\
\text{Period-} P \text{ oscillation} \quad \text{where } P \in (1, R] \text{ and } \text{mod}(R, P) = 0 \quad \text{if } \text{mod}(\tau, R) \neq 0
\end{cases}
\]

In the following examples \( dt \) refers to numerical simulation time size.
Example 1: Consider parameters \( i(t) = 2 + \sin(t), \tau = 4\pi, SL(0) = 0 \) with \( dt = 0.001 \). They satisfy \( \text{mod}(4\pi, 2\pi) = 0 \) therefore stock variable must exhibit goal seeking behavior while outflow must display periodic motion as confirmed in Figure C.4.

![Figure C.4. Periodic inflow discrete delay atomic structure, \( \tau = 4\pi \)](image)

Example 2: We use the parameter values of Example 1, but only change \( dt \) to 0.1 (100 times larger interval). Parameters satisfy \( \text{mod}(4\pi, 2\pi) = 0 \) therefore stock variable must exhibit goal seeking behavior while outflow must display periodic motion as confirmed in Figure C.5. Although we increased simulation time step 100 times, results do not change (i.e. behavior is insensitive to \( dt \)) which is a property of continuous time systems.

![Figure C.5. Periodic inflow discrete delay atomic structure, \( \tau = 4\pi, \ dt = 0.1 \)](image)
Example 3: Consider parameters $i(t) = 3/2 + \sin(t)$, $\tau = 5\pi/2$, $SL(0) = 0$ with $dt = 0.001$. They satisfy $\text{mod}(5\pi/2, 2\pi) \neq 0$ therefore both stock and outflow variables must display same periodic behavior as inflow as confirmed in Figure C.6.

![Figure C.6. Periodic inflow discrete delay atomic structure, $\tau = 5\pi/2$](image)

Example 4: Consider parameters $i(t) = 1 + \cos(t)$, $\tau = 3\pi/2$, $SL(0) = 0$ with $dt = 0.001$. They satisfy $\text{mod}(3\pi/2, 2\pi) \neq 0$ therefore both stock and outflow variables must display same periodic behavior as inflow as confirmed in Figure C.7.

![Figure C.7. Periodic inflow discrete delay atomic structure, $\tau = 3\pi/2$](image)
Example 5: We use the parameter values of Example 4, but only change \( dt \) to 0.25 (250 times larger interval). Parameters satisfy \( \text{mod}(3\pi/2, 2\pi) \neq 0 \) therefore both stock and outflow variables must display same periodic behavior as inflow as confirmed in Figure C.8. Although we increased simulation time step 250 times, results do not change (i.e. behavior is insensitive to \( dt \)) which is a property of continuous time systems.

![Figure C.8. Periodic inflow discrete delay atomic structure, \( \tau = 3\pi/2, dt = 0.25 \)](image)

C.3. Inventory Model with Continuous Order Policy

We apply results of Section C.1 (equilibrium and stability analysis) and Section C.2 (simple atomic structure analysis) to the following inventory model with continuous order policy.

![Figure C.9. Stock flow representation with one goal](image)
System equations are as follows:

\[ I'(t) = f(t) - D \]
\[ SL'(t) = O(t) - f(t) \]
\[ f(t) = O(t - \tau) \]
\[ O(t) = \frac{I^* - I(t)}{AT} + D \tag{C.9} \]

where \( SL(t) \), \( I(t) \), \( O(t) \), \( f(t) \) represent supply line, inventory, order and receiving rate at time \( t \), \( I^* \) is desired inventory and \( AT \) is adjustment time.

In order to carry out equilibrium analysis plug zero to delay \( \tau \) in Equation (C.9).

\[ I'(t) = f(t) - D = O(t) - D = \frac{I^* - I(t)}{AT} = 0 \Rightarrow I_e = I^* \tag{C.10} \]
\[ SL'(t) = O(t) - f(t) = O(t) - O(t) = 0 \]

From Equation (C.10), equilibrium point of inventory is \( I^* \) while supply line equilibrium can not be obtained from standard equilibrium analysis techniques. We try to find \( SL_e \) using results of Section C.2.

\[ I'(t) = f(t) - D = 0 \Rightarrow f_e = D \]

Although \( f(t) = O(t - \tau) = \frac{I^* - I(t - \tau)}{AT} + D \) variable at the beginning, eventually it settles down to \( f_e = D \). Therefore \( SL_e = \tau D \) from Equation (C.6).

For stability analysis first plug \( f(t) \) and \( O(t) \) equations in \( I(t) \) and \( SL(t) \) equations in Equation (C.9).
\[ I'(t) = \frac{I^* - I(t-\tau)}{AT} + D - D = \frac{I^* - I(t-\tau)}{AT} \]

\[ SL'(t) = O(t) - f(t) = \frac{I^* - I(t)}{AT} + D - \frac{I^* - I(t-\tau)}{AT} - D = \frac{I(t-\tau) - I(t)}{AT} \]  \hspace{1cm} (C.11)

From Equation (C.11), if equilibrium of inventory is shown to be stable, system is stable since supply is a sum of differences and if behavior of inventory is stable that difference goes to zero and therefore sum goes to a finite number.

Inventory equation is a pure delay differential equation, compare with Equations (C.1) and (C.12).

\[ \frac{dI(t)}{dt} = -\frac{I(t-\tau)}{AT} + \frac{I^*}{AT} \]  \hspace{1cm} (C.12)

Using Equation (C.2), if the following condition is satisfied behavior of inventory and thus the system is stable.

\[ 0 < \tau/AT < \pi/2 \]  \hspace{1cm} (C.13)

In Figure C.10, behavior of inventory is shown for different parameter values of \( AT \), for all parameter values \( \tau/AT \) ratio is in the interval \( (0, \pi/2) \) therefore all four behaviors are stable and as the ratio \( \tau/AT \) decreases, behavior becomes more stable.

In Figure C.11, \( \tau/AT \) ratio is at the border value \( \pi/2 \), therefore both supply line and inventory display borderline stable oscillatory behavior.

In Figure C.12, behavior of inventory is shown for different parameters values of \( AT \), for all parameter values \( \tau/AT \) ratio is outside the interval \( (0, \pi/2) \) therefore all four behaviors are unstable and as the ratio \( \tau/AT \) increases, behavior becomes more unstable.
Figure C.10. Inventory dynamics ($\tau = \pi$ and $AT = 3, 5, 7, 9$ for 1st, 2nd, 3rd, and 4th runs)

Figure C.11. Inventory and supply line dynamics $\tau = \pi$ and $AT = 2$
Figure C.12. Inventory dynamics ($\tau = \pi$ and $AT = 1.5, 1.4, 1.3$ for 1st, 2nd and 3rd runs)
APPENDIX D: SAMPLE MODEL AND EQUATIONS

The following is the stock flow diagram of \((s, S)\) policy with second order continuous delay.

![Diagram](image)

Figure D.1. Stock flow representation of \((s, S)\) policy, 2\textsuperscript{nd} order continuous delay

**DECISION/POLICY RULE**

decision_rule = if(effInv<=orderPoint) then orderuptoLevel-E else 0

effInv = Inventory+Supply_line1+Supply_line2

**GOAL FORMATION**

DAVGL = demand*totDelay

DMINV = demand
orderPoint = DAVGSL + DMINV
orderuptoLevel = orderPoint + Q
Q = 75

STOCK ACQUISITION SYSTEM
Inventory(t) = Inventory(t - dt) + (flow2 - delivery) * dt
INIT Inventory = 0

INFLOWS:
flow2 = Supply_line2/delay2_

OUTFLOWS:
delivery = demand
Supply_line1(t) = Supply_line1(t - dt) + (order - flow1) * dt
INIT Supply_line1 = 0

INFLOWS:
order = decision_rule

OUTFLOWS:
flow1 = Supply_line1/delay1
Supply_line2(t) = Supply_line2(t - dt) + (flow1 - flow2) * dt
INIT Supply_line2 = 0

INFLOWS:
flow1 = Supply_line1/delay1

OUTFLOWS:
flow2 = Supply_line2/delay2_
delay1 = 2
delay2_ = 4
demand = 20
totDelay = delay1 + delay2
REFERENCES


