Symmetries of graphs

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Workshop on Graph Theory and Its Applications VI

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Two minicourses led by:
Endre Boros (Rutgers University, USA), and
Robin Wilson (Open University, United Kingdom).
All questions should be sent to sygn@upr.si
Overview

- Graph isomorphism problem
- Automorphism group of a graph
- Asymmetric graphs
- Graphs with large degree of symmetry
A graph is an ordered pair $X = (V, E)$, where $V$ denotes the set of vertices, and $E$ denotes the set of edges of the graph $X$.

$V = \{1, 2, 3, 4, 5\}$;

$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{2, 4\}, \{4, 5\}, \{1, 5\}\}$

We consider only finite, simple, undirected graphs.
What is the difference between the two graphs below?

For such graphs we say that they are ISOMORPHIC.
Isomorphic graphs

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Only in the name of the vertices.
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Isomorphic graphs

Definition

We say that graphs $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$ are **ISOMORPHIC** if there exists a **bijective** function $f : V_1 \to V_2$, such that

$$(\forall u, v \in V_1) \{u, v\} \in E_1 \iff \{f(u), f(v)\} \in E_2.$$ 

We write $X_1 \cong X_2$. Function $f$ is called **ISOMORPHISM**.
Isomorphic graphs

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$$(\forall u, v \in V_1) \{u, v\} \in E_1 \Leftrightarrow \{f(u), f(v)\} \in E_2.$$

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Function $f : \{1, 2, 3, 4\} \rightarrow \{A, B, C, D\}$, defined with $f(1) = B$, $f(2) = A$, $f(3) = D$, $f(4) = C$ is isomorphism.
Complement of a graph

**Definition**

Let $X = (V, E)$ be a graph. A **COMPLEMENT** of the graph $X$, is the graph $\overline{X} = (V, \overline{E})$, where for $u, v \in V (u \neq v)$

$$\{u, v\} \in \overline{E} \iff \{u, v\} \notin E.$$
Complement of a graph

Definition
Let $X = (V, E)$ be a graph. A COMPLEMENT of the graph $X$, is the graph $X' = (V, E')$, where for $u, v \in V (u \neq v)$

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Theorem
Two graphs $X$ and $Y$ are isomorphic if and only if their complements $X'$ and $Y'$ are isomorphic.
Are the following two graphs isomorphic?
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We look at the complements.
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At the same time, isomorphism for many special classes of graphs can be solved in polynomial time, and in practice graph isomorphism can often be solved efficiently (for example trees, planar graphs, graphs of bounded degree, graphs of bounded treewidth...).
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In 2015, Babai announced a quasipolynomial time algorithm for all graphs, that is, one with running time $2^{O((\log n)^c)}$ for some fixed $c > 0$. 
Problem

Determine the number of non-isomorphic graphs with $n$ vertices.
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\( n = 2, \ g_n = 2 \)
Enumerating non-isomorphic graphs

Problem

*Determine the number of non-isomorphic graphs with $n$ vertices.*

Let $g_n$ denote the number of non-isomorphic graphs of order $n$.

$n = 2, \quad g_n = 2$

- \begin{align*}
  & 1 \quad 2 \\
  & 3 \quad 2
\end{align*}

$n = 3, \quad g_n = 4$

- \begin{align*}
  & 1 \\
  & 3 \quad 2
\end{align*}

- \begin{align*}
  & 1 \quad 1 \\
  & 3 \quad 2
\end{align*}

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Enumerating non-isomorphic graphs

\[ n = 4, \quad g_n = 11 \]

![Graphs](image)
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With $S_n$ we denote the group of all permutations of the set $\{1, 2, \ldots, n\}$. Observe that $|S_n| = n!$.
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$$g = (1\ 2\ 3\ 4\ 5\ 6) = (1)(2\ 5\ 3)(4\ 6).$$
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1 + 1 + 2
1 + 1 + 1 + 1
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4 &= (0, 0, 0, 1) \\
1 + 3 &= (1, 0, 1, 0) \\
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Observe that $\sum_{k=1}^{n} kc_k = n$, and a permutation $g \in S_n$ can be associated with the vector $(c_1(g), \ldots, c_n(g))$. 

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**Enumerating non-isomorphic graphs**

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---

**Ademir Hujdurović**  
**Symmetries of graphs**
For a given partition \((c) = (c_1, c_2, \ldots, c_n)\) of a positive integer \(n\) let

\[
\gamma(c) = \sum_k \left\lfloor \frac{k}{2} \right\rfloor c_k + \sum_k \frac{k c_k (c_k - 1)}{2} + \sum_{r < t} \gcd(r, t) c_r c_t.
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Enumerating non-isomorphic graphs

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Theorem (Polya, 1951)

The number of non-isomorphic graphs of order \(n\) equals

\[
g_n = \sum_{(c) = (c_1, \ldots, c_n)} \frac{2^\gamma(c)}{\prod k^{c_k} c_k!},
\]

where the sum goes throughout all the partitions \((c)\) of \(n\).
## Enumerating non-isomorphic graphs

<table>
<thead>
<tr>
<th>$n$</th>
<th>Non-isomorphic graphs of order $n$</th>
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Let $\Omega = \{\{i, j\} \mid 1 \leq i < j \leq n\}$. (We can think of $\Omega$ as of the edge set of the complete graph on $n$ vertices).
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Let $X = (V, E)$ be an arbitrary graph with vertex set $V$. Then $E \subseteq \Omega$. Hence we can represent every graph with vertex set $V$ uniquely as the subset of $\Omega$, that is, there exists a bijection between the set of all graph on $n$ vertices and the set $\mathcal{P}(\Omega)$. 
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Every permutation \( g \in S_n \) induces a permutation of the set \( \Omega \). The corresponding permutation of \( \Omega \) is denoted by \( g^{(2)} \) and the corresponding permutation group is denoted by \( S_n^{(2)} \).
Two graphs $X_1 = (V, E_1)$ and $X_2 = (V, E_2)$ are isomorphic if and only if there exists $g \in S_n$ such that $g^{(2)}(E_1) = E_2$. 

Therefore, the number of non-isomorphic graphs of order $n$ is equal to the number of orbits in the action of $S_n$ on $P(\Omega)$.

Theorem (Orbit-counting theorem (also called Cauchy-Frobenius or Burnside's))

Let $G$ be a group acting on a set $\Delta$. Then the number of orbits equals

$$1 |G| \sum_{g \in G} |\text{Fix} \Delta(g)|.$$ 

Here $\text{Fix} \Delta(g) = \{x \in \Delta | g(x) = x\}$. 

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Theorem (Orbit-counting theorem (also called Cauchy-Frobenius or Burnside's))

Let $G$ be a group acting on a set $\Delta$. Then the number of orbits equals $\frac{1}{|G|} \sum_{g \in G} |\text{Fix}\, \Delta(g)|$.

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**Theorem (Orbit-counting theorem (also called Cauchy-Frobenius or Burnside’s))**

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$$\frac{1}{|G|} \sum_{g \in G} |Fix_{\Delta}(g)|.$$  

Here $Fix_{\Delta}(g) = \{x \in \Delta \mid g(x) = x\}$. 
It follows that the number of non-isomorphic graphs of order $n$ equals

\[ \frac{1}{|S_n^{(2)}|} \sum_{g \in S_n^{(2)}} |\text{Fix}_{\mathcal{P}(\Omega)}(g)| = \frac{1}{n!} \sum_{g \in S_n^{(2)}} |\text{Fix}_{\mathcal{P}(\Omega)}(g)|. \]
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How to determine $|\text{Fix}_P(\Omega)(g^{(2)})|$?

Example: Let $n = 4$, $g \in S_4$ given with $g = (1 \ 2)(3 \ 4)$. 

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Enumerating non-isomorphic graphs

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$$g^{(2)} = (\{1, 2\})(\{3, 4\})(\{1, 3\} \{2, 4\})(\{1, 4\} \{2, 3\}).$$
It follows that the number of non-isomorphic graphs of order $n$ equals

$$\frac{1}{|S_n^{(2)}|} \sum_{g \in S_n^{(2)}} |Fix_{\mathcal{P}(\Omega)}(g)| = \frac{1}{n!} \sum_{g \in S_n^{(2)}} |Fix_{\mathcal{P}(\Omega)}(g)|.$$

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If $E \subseteq \Omega$ is fixed by $g^{(2)}$, then one cycle in the cyclic decomposition of $g^{(2)}$ must be either completely contained in $E$, or completely outside of $E$. Hence $|Fix_{\mathcal{P}(\Omega)}(g)| = 2^4 = 16.$
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How to determine \( |\text{Fix}_{\mathcal{P}(\Omega)}(g^{(2)})| \)?

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If \( E \subseteq \Omega \) is fixed by \( g^{(2)} \), then one cycle in the cyclic decomposition of \( g^{(2)} \) must be either completely contained in \( E \), or completely outside of \( E \). Hence \( |\text{Fix}_{\mathcal{P}(\Omega)}(g)| = 2^4 = 16 \).

In general \( |\text{Fix}_{\mathcal{P}(\Omega)}(g)| = 2^c \), where \( c \) is the number of cycles in cyclic decomposition of \( g^{(2)} \).
Lemma

If \( g \in S_n \) is a permutation that corresponds to the partition \( (c) = (c_1, c_2, \ldots, c_n) \) (that is \( g \) has \( c_k \) cycles of length \( k \)) then there are \( \gamma(c) \) cycles in the cyclic decomposition of \( g^{(2)} \), where

\[
\gamma(c) = \sum_{k} \left\lfloor \frac{k}{2} \right\rfloor c_k + \sum_{k} \frac{k c_k (c_k - 1)}{2} + \sum_{r<t} \gcd(r, t)c_r c_t.
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Theorem (Polya, 1951)

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g_n = \sum_{(c) = (c_1, \ldots, c_n)} \frac{2^{\gamma(c)}}{\prod k^{c_k} c_k!},
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where the sum goes throughout all the partitions $(c)$ of $n$. 

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Symmetries of graphs
Asymptotic formulas for the number of non-isomorphic graphs

Polya obtained the following formula for the asymptotic number of graphs $g_n$ of order $n$:

**Theorem (Polya)**
The number of graphs of order $n$ asymptotically equals to $2^{(n^2)} n!$.

Oberschelp in 1967 obtained a better approximation.

**Theorem (Oberschelp)**
$$g_n = 2^{(n^2)} n! \left(1 - \frac{2}{n^2} - \frac{n}{n^2} + \frac{8}{n^2} \left(\frac{3}{n^2} - \frac{7}{3}\right) \left(\frac{3}{n^2} - \frac{9}{3}\right) n^5 + O\left(\frac{n^5}{n^2}\right)\right).$$
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$$g_n = \frac{2^{\binom{n}{2}}}{n!} \left(1 - 2\frac{n^2 - n}{2^n} + \frac{8n!(3n - 7)(3n - 9)}{(n - 4)!2^{2n}} + O\left(\frac{n^5}{2^{5n/2}}\right)\right),$$
Definition

Let $X$ be a graph. An **AUTOMORPHISM** of a graph $X$ is isomorphism from $X$ to $X$. In other words, automorphism is a permutation $f$ of a vertex set of $X$, that preserves the edge set of $X$, that is

$$\{u, v\} \in E \iff \{f(u), f(v)\} \in E.$$ 

With $Aut(X)$ we denote the set of all automorphism of a graph $X$. 
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The set $Aut(X)$ together with the operation $\circ$ (composition of functions) is a group, and is called the **automorphism group** of $X$. 
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If a graph $X$ has $n$ vertices, then $Aut(X) \leq S_n$.
Note that automorphism preserves degree, distance between vertices etc.
Find all automorphisms of the graph below.

\[ g_1 = \text{id} \text{ and } g_2 = (1 4) \].
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1
\[ \begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\downarrow \\
4 \\
\end{array} \]

\( g_1 = id \) and \( g_2 = (1 \ 4) \).
Find all automorphisms of the graph below.

\[
\begin{align*}
\rho &= (1 \ 2 \ 3 \ 4 \ 5 \ 6) \\
\tau &= (1)(2\ 6)(3\ 5)(4)
\end{align*}
\]

\[\text{Aut}(C_6) = \langle \rho, \tau \rangle \cong D_{12}.\]

\[|\text{Aut}(C_6)| = 12.\]
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Asymmetric graphs

Definition

A graph $X$ with at least two vertices such that $\text{Aut}(X) = \{id\}$ is called **ASYMMETRIC**.
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Prove that the smallest asymmetric graph has 6 vertices.
Theorem

Almost all finite graphs are asymmetric, that is the ratio between the number of asymmetric graphs of order \( n \) and all graphs of order \( n \) tends to 1 as \( n \) grows.

This follows from the fact that the number of graphs of order \( n \) is asymptotically equal to \( \frac{2\binom{n}{2}}{n!} \).

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Can any given group be automorphism group of some graph?
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Can any given group be automorphism group of some graph?

Theorem (Frucht, 1939)

For a given finite group \(G\), there exists infinitely many connected graphs \(X\) such that \(\text{Aut}(X) \cong G\).
Probably the best studied class of graphs having large automorphism group is the class of vertex-transitive graphs.
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Examples of vertex-transitive graphs

**Definition**

Let $n$, $k$, and $i$ be fixed positive integers, with $n > k > i$ let $S$ be a fixed set of size $n$; and define $J(n, k, i)$ as follows. The vertices of $J(n, k, i)$ are the subsets of $S$ with size $k$, where two subsets are adjacent if their intersection has size $i$. Therefore, $J(n, k, i)$ has $\binom{n}{k}$ vertices.

For $n > 2k$, the graphs $J(n, k, k-1)$ are known as the Johnson graphs, and the graphs $J(n, k, 0)$ are known as the Kneser graphs.

$J(5, 2, 0)$ is the Petersen graph.

**Theorem**

Graph $J(n, k, i)$ is vertex-transitive, for all values of $n$, $k$, and $i$. 

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Symmetries of graphs
Definition

Let \( n, k, \) and \( i \) be fixed positive integers, with \( n > k > i \) let \( S \) be a fixed set of size \( n \); and define \( J(n, k, i) \) as follows. The vertices of \( J(n, k, i) \) are the subsets of \( S \) with size \( k \), where two subsets are adjacent if their intersection has size \( i \). Therefore, \( J(n, k, i) \) has \( \binom{n}{k} \) vertices.

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$X = (V, E)$-graph and $v$ a vertex of $X$.

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Let \( X = (V, E) \) be a vertex-transitive graph, and let \( u, v \in V \). Then there exists \( f \in Aut(X) \) such that \( f(u) = v \). Since automorphisms preserve edges, it follows that \( f(N_X(u)) = N_X(v) \).
## Definition

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## Theorem

*Every vertex-transitive graph is regular.*

## Proof.

Let $X = (V, E)$ be a vertex-transitive graph, and let $u, v \in V$. Then there exists $f \in Aut(X)$ such that $f(u) = v$. Since automorphisms preserve edges, it follows that $f(N_X(u)) = N_X(v)$. Since $f$ is bijective mapping, it follows that $|f(N_X(u))| = |N_X(u)|$. Hence $d_X(u) = d_X(v)$.
Is every regular graph vertex-transitive?
Properties of vertex-transitive graphs

Is every regular graph vertex-transitive? \textbf{NO.}

\textbf{Figure:} Frucht Graph is 3-regular asymmetric graph
Let $n$ be a positive integer, and let $S \subseteq \{1, \ldots, n-1\}$ be a symmetric subset of $\{1, \ldots, n\}$, that is $S = -S$. A CIRCULANT graph $\text{Circ}(n; S)$ is the graph with vertex set $V = \{0, 1, \ldots, n-1\}$ and edge set

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*Figure: $\text{Circ}(8, \{1, -1, 3, -3\})$*
Theorem

*Every circulant is vertex-transitive.*
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*Every circulant is vertex-transitive.*

Proof.

Let $X = \text{Circ}(n, S)$ be an arbitrary circulant. Let

$$\rho = (0 \ 1 \ 2 \ \ldots \ n-1),$$

that is $\rho(x) = x + 1$, for every $x \in \mathbb{Z}_n$. 

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Every circulant is vertex-transitive.

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Let $X = \text{Circ}(n, S)$ be an arbitrary circulant. Let $\rho = (0 \ 1 \ 2 \ \ldots \ n-1)$, that is $\rho(x) = x + 1$, for every $x \in \mathbb{Z}_n$. We claim that $\rho \in \text{Aut}(X)$. 
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Properties of circulants:

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Properties of circulants:

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3. There is a polynomial time recognition algorithm for circulants;
4. Isomorphism problem for circulant graphs can be solved in polynomial time.
The concept of circulant graphs can be generalized to the concept of Cayley graphs.

**Definition**

Let $G$ be a finite group, and let $S$ subset of $G$, such that $S = S^{-1}$ and $1_G \not\in S$. A **CAYLEY** graph $\text{Cay}(G, S)$ is defined as the graph with vertex set $V = G$ and edge set

$$E = \{\{x, x \cdot s\} \mid x \in G\}.$$
Example

\[ G = \{1, a, b, c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \ (a^2 = b^2 = c^2 = 1, \ ab = ba = c, \ ac = ca = b, \ bc = cb = a), \]
\[ S = \{b, c\} \]

Figure: \( \text{Cay}(G, \{b, c\}) \)
Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and let $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.
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To which graph is $\text{Cay}(G, S)$ isomorphic?

To the complete bipartite graph with parts of size $n! / 2$. 

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Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and let $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Let $G = S_n$, and let $S$ be the set of all transpositions. To which graph is $\text{Cay}(G, S)$ isomorphic?

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Let $G = S_n$, and let $S$ be the set of all transpositions. To which graph is $\text{Cay}(G, S)$ isomorphic?
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\[ \text{Diagram showing a cube with vertices labeled (0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1).} \]
Theorem

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Proof.
Let \( X = Cay(G, S) \), and let \( g \in G \) be arbitrary.
Theorem

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Proof.

Let $X = \text{Cay}(G, S)$, and let $g \in G$ be arbitrary. Define mapping $g_L : G \rightarrow G$ with $g_L(x) = g \cdot x$, for all $x \in G$. It is easy to see that $g_L$ is bijective mapping. Also, for an edge $\{x, xs\}$ its image under $g_L$ is $\{gx, gxs\}$ which is again an edge in $\text{Cay}(G, S)$. Hence $g_L \in \text{Aut}(X)$, for every $g \in G$. Let $x$ and $y$ be two arbitrary vertices in $X$. Let $g = yx^{-1} \in G$. Then $g_L(x) = gx = (yx^{-1})x = y$, hence $X$ is vertex-transitive.
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Cayley graphs

If $G = \mathbb{Z}_n$, then we get a circulant graph, hence circulants are Cayley graphs on cyclic groups.
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Is every vertex-transitive graph Cayley?
Cayley graphs

If $G = \mathbb{Z}_n$, then we get a circulant graph, hence circulants are Cayley graphs on cyclic groups.

Is every vertex-transitive graph Cayley? NO.

Figure: Petersen graph is vertex-transitive but not Cayley
Definition

For a positive integer $n$, we say that $n$ is a **CAYLEY NUMBER**, if every vertex-transitive graph of order $n$ is a Cayley graph. Otherwise, $n$ is said to be a **NON-CAYLEY NUMBER**.
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Since Petersen graph is vertex-transitive but not Cayley graph, and it has 10 vertices, it follows that 10 is a non-Cayley number.
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Problem (Marušič, 1983)

*Characterize Cayley (non-Cayley) numbers.*
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**Problem (Marušič, 1983)**

*Characterize Cayley (non-Cayley) numbers.*

If \( n \) is a non-Cayley number, then any multiple \( kn \) of \( n \) is also a non-Cayley number. Namely, take \( k \) copies of a non-Cayley vertex transitive graph of order \( n \).
Cayley numbers

What is known?

Ca\textsuperscript{ley} numb\textsuperscript{ers}

Numbers, \(p, p\textsuperscript{2}, p\textsuperscript{3}\) are Cayley numbers (Maru\text-superscript{a}€, 1983);

Numbers divisible by \(p\textsuperscript{4}\) are non-Cayley; (Maru\text-superscript{a}€, 1983);

\(2p\) is a Cayley number when \(p \equiv 3 \) (mode 4) (Alspach and Sutclie, 1979);

Numbers divisible by \(p\textsuperscript{2}\) are non-Cayley except for \(n = p\textsuperscript{2}\), \(n = p\textsuperscript{3}\) or \(n = 12\) (McKa\text-y and Praeger, 1996);

Cayley numbers that are a product of two distinct primes are classied (McKa\text-y and Praeger, 1996);

Cayley numbers that are products of three distinct primes are classied (Iranmanesh and Praeger, 2001);

There exists an infinite set \(S\) of primes such that every finite product of distinct elements from \(S\) is a Cayley number (Dobson and Spiga, 2016).
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- Cayley numbers that are a product of two distinct primes are classified (McKay and Praeger, 1996);
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Cayley numbers

What is known?

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Is it possible to enumerate vertex-transitive graphs of order $n$?

The only known result for enumeration of vertex-transitive graphs of order $n$ is when $n = p$ is a prime.

**Theorem (Turner, 1967)**

The number of vertex-transitive graphs with prime number of vertices $p$, is equal to

$$
\sum_{d | m} \phi(d) \cdot 2^{m/d},
$$

where $m = \left( p - 1 \right) / 2$ and $\phi$ is the Euler's totient function.

**Proof idea:** Every vertex-transitive graph of prime order is in fact a circulant $\text{Circ}(p, S)$.

Turner also proved two circulants of prime order $\text{Circ}(p, S_1)$ and $\text{Circ}(p, S_2)$ are isomorphic if and only if there exists $k \in \mathbb{Z}^*$ such that $kS_1 = S_2$.

Therefore, in order to count the number of vertex-transitive graphs of order $p$, it suffices to count the number of orbits in the action of $\mathbb{Z}^*_p$ on the subsets of $\mathbb{Z}_p$. 

Ademir Hujdurović

Symmetries of graphs
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Vertex-transitive graphs

<table>
<thead>
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<tr>
<td>16</td>
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<td>31</td>
<td>2191</td>
</tr>
</tbody>
</table>

**Table:** Number of connected vertex-transitive graphs of order $n \leq 31$. 
Definition

For a connected graph $X$, the **EDGE CONNECTIVITY** is the minimum number of edges such that the graph obtained after deleting those edges is disconnected, and will be denoted by $\kappa_1(X)$.
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It is clear that for every graph $X$, it holds $\kappa_1(X) \leq \delta(X)$, where $\delta(X)$ is the minimum degree in $X$. 
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Theorem
*If $X$ is a connected vertex-transitive graph, then its edge connectivity is equal to its degree.*
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Theorem

Let $X$ be a connected vertex-transitive graph with degree $d$. Then $\kappa(X) \geq 2d + 1$. 

Ademir Hujdurović  
Symmetries of graphs
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Let $X$ be a connected vertex-transitive graph with degree $d$. Then $\kappa(X) \geq 2^{d+1}$. 

Ademir Hujdurović  Symmetries of graphs
Definition

A MATCHING $M$ in a graph $X$ is a set of edges such that no two have a vertex in common. A matching that covers every vertex of $X$ is called a PERFECT MATCHING or a 1-factor. A MAXIMUM MATCHING is a matching with the maximum possible number of edges.
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Theorem
Let \( X \) be a connected vertex-transitive graph. Then \( X \) has a matching that misses at most one vertex, and each edge is contained in a maximum matching.
Hamiltonicity of VT graphs

Definition

A HAMILTON PATH in a graph is a path that meets every vertex, and a HAMILTON CYCLE is a cycle that meets every vertex.
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Every connected vertex-transitive graph contains a Hamilton path.
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Every connected vertex-transitive graph contains a Hamilton path.

There are only 5 known vertex-transitive graphs without Hamilton cycle. They are

- $K_2$;
- Petersen graph;
- Truncated Petersen graph;
- Coxeter graph;
- Truncated Coxeter graph.
Truncated Petersen graph
Except for $K_2$, none of the five known vertex-transitive graphs without hamilton cycle is Cayley graph. This has lead to the following "Folklore" conjecture.
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**Conjecture**

*Every connected Cayley graph of order at least 3 contains a Hamilton cycle.*
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**Conjecture**

*Every connected Cayley graph of order at least 3 contains a Hamilton cycle.*

**Conjecture (Thomasen)**

*There are finitely many connected vertex-transitive graphs without Hamilton cycle.*

**Conjecture (Babai)**

*There are infinitely many connected vertex-transitive graphs (or Cayley graphs) without Hamilton cycle.*
Theorem (Chen and Quimpo, 1981)

Every connected Cayley graph on abelian group of order at least 3 contains a Hamilton cycle.
Hamiltonicity of VT graphs

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To prove this, we need to define Cartesian product of graphs.

Definition

The Cartesian product $X \square Y$ of two graphs $X$ and $Y$ is the graph with vertex set $V(X) \times V(Y)$ and two vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)$ or $y_1 = y_2$ and $\{x_1, x_2\} \in E(X)$. 

Exercise

Prove that the Cartesian product of two vertex-transitive (resp. Cayley) graphs is vertex-transitive (resp. Cayley) graph.
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If $|S| = 2$, then either $S = \{a, a^{-1}\}$ or $S = \{a, b\}$ with $a^{-1} = a$, $b^{-1} = b$. In the first case, we have a Hamilton cycle $1 - a - a^2 - \ldots$. 
Let $X = \text{Cay}(G, S)$, where $G$ is abelian group, $|G| \geq 3$ and $\langle S \rangle = G$. Proof is by induction on $|S|$. If $|S| = 1$, then the result trivially follows.

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Ademir Hujdurović  
Symmetries of graphs
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If $|S| \geq 3$, then we define $T = S \setminus \{a, a^{-1}\}$ for some $a \in S$. Let $M = \langle T \rangle$. The subgraph of $X$ induced by $M$ is isomorphic to $\text{Cay}(M, T)$, which contains hamiltonian cycle by induction hypothesis. Using the fact that $G$ is abelian, one can see that $X$ has a subgraph isomorphic to the Cartesian product $\text{Cay}(M, T) \square P_k$, where $k$ is the smallest integer such that $a^k \in M$. 
Definition

Dihedral group of order $2n$ is

$$D_{2n} = \langle \rho, \tau \mid \rho^n = \tau^2 = 1, \tau \rho = \rho^{-1} \tau \rangle.$$
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*Every connected cubic (that is 3-regular) Cayley graph on dihedral group contains a Hamilton cycle.*
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Problem

Does every connected Cayley graph on dihedral group $D_{2n}$ contain a Hamilton cycle?
Theorem (Witte, 1986)

Every connected Cayley graph on a $p$-group contains a Hamilton cycle.

Theorem (Keating and Witte, 1985)

Every connected Cayley graph of a group with a cyclic commutator subgroup of prime power order, has a Hamilton cycle.

Theorem (Alspach, Chen and Dean, 2010)

Every connected Cayley graph on a generalized Dihedral group whose order is divisible by 4 contains a Hamilton cycle.
Hamiltonicity of VT graphs

Let $p$ denote a prime.

**Theorem**

Let $X$ be a connected vertex-transitive graph of order $p$. Then $X$ contains a Hamilton cycle.

**Theorem (Alspach, 1979)**

Every connected vertex-transitive graph of order $2p$ contains a Hamilton cycle, unless it is isomorphic to the Petersen graph.

**Theorem (Marušič, 1985)**

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Theorem (Zhang, 2015)
Vertex-transitive graphs of order $p^5$ contain a Hamilton cycle.
Thank you!!!