The Archimedean Assumption in Fuzzy Set Theory *

Taner Bilgiç
Department of Industrial Engineering
University of Toronto
Toronto, Ontario, M5S 1A4 Canada
taner@ie.utoronto.ca

Abstract

The Archimedean axiom in fuzzy set theory is critically discussed. The axiom is brought into perspective within a measurement theoretic framework and then its validity for fuzzy set theory is questioned. The discussion sheds light into what type of vagueness fuzzy set theory models.

1 Introduction

It is one of the basic tenets of fuzzy set theory to take into account the *continuous* degrees of membership. In that way, fuzzy set theory is distinguished from other many-valued logics.

Continuous membership functions and continuous Archimedean triangular norms and conorms together with a negation operator describe an algebraic structure that defines fuzzy set theory.

We have investigated the semantic issues that such an algebraic structure raises elsewhere [1, 4, 2, 3] from a measurement point of view. In this paper, we concentrate on the Archimedean axiom and discuss its relevance to fuzzy set theory. In particular we are trying to answer the following questions:

1. Is the Archimedean axiom relevant for fuzzy set theory?

2. What does one gain or loose when one adopts the Archimedean axiom?

Once again we do this in the framework provided by measurement theory. First, we investigate the use of the Archimedean axiom in other contexts —particularly subjective probability theory and utility theory— and then investigate the implications of the Archimedean axiom in fuzzy set theory.

Figure 1 shows how measurement theory works: the qualitative structure corresponds to a conceptual schema and the numerical structure is the corresponding representation. Notice that the representation is "two-way", i.e., whatever happens in the qualitative structure must be mirrored in the numerical structure and vice versa.

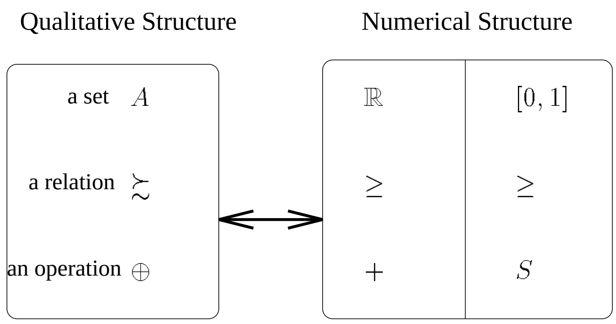

Figure 1. Representation in Measurement Theory

We emphasize that it is this *reverse* implication which makes Archimedean axiom a problem. Archimedean axiom is a property of the numerical structure that *has to be* reflected on the qualitative structure. We are going to argue that this is not always possible.

At least since Goguen [7] we know that membership functions can be represented on weaker structures than the unit interval. All that is required is that the structure be a Heyting algebra. However, fuzzy set theorists continue to use the unit interval, probably for its ease in conveying the graded membership concept to non-specialists using the familiar real numbers. We are investigating the consequences of that assumption.

2 Archimedean axiom in general

The Archimedean axiom has been given that name since it corresponds to the Archimedean property of real numbers [9]: for any positive number x no matter how small and another positive number y no matter how large, there exists a positive integer n such that $nx > y$. This implies that two real numbers are *comparable* and their ratio is *not infinite*. Another way to state this axiom is: for any real numbers x and y, the set of integers $\{n : y > nx\}$ is a finite set.

* Appeared in Proceedings of NAFIPS'96 June 19-22, 1996, University of California, Berkeley.

In algebraic structures, Archimedean axioms are expressed in many forms. But the idea is always to represent this (nice) property of real numbers. However, Archimedean axiom is only necessary when we want a mapping from the algebraic structure into *real numbers* (a real representation).

If one does not require "the comfort" of real numbers at the outset, Archimedean axiom is not necessary. From a practical point of view, it is very hard, if not impossible, to test for empirical meaningfulness of the Archimedean axiom in fuzzy sets. We give some insights as to *why* testing for empirical meaningfulness is important.

Note that, in a *finite* structure, Archimedean axiom trivially holds and therefore it is not necessary for the real representation. Furthermore, the Archimedean axiom cannot be expressed in an elementary language [10]. In an elementary formal language, the Archimedean axiom can be stated as statement with countably infinite disjunctions :

$$\forall(a)\forall(b)[a^{(1)} \succ b \vee a^{(2)} \succ b \vee a^{(3)} \succ b \vee \cdots]$$

where $a^{(n)} \succ b$ is an abbreviation for the first order statement that n concatenations of a with itself exceeds b.

2.1 Probability

In the axiomatization of qualitative probability [9, 5], Archimedean axiom together with the necessary conditions on the qualitative structure is not sufficient to come up with a unique representation for probability measures (see [9, Section 5.2.1] for example). Therefore, usually another strong axiom, namely the existence of a fine partition to model pure randomness as the uniform distribution, is postulated. Once this is done, which endows the structure with a fine grain structure that is akin to real numbers, then the Archimedean axiom is accepted on normative grounds (because it exists in the representation of the continuum as we know it!).

2.2 Utility

In utility theory, Archimedean axiom arises in two manners. In the "naive" utility theory (or value theory) it has the structure exactly like the one we are going use in fuzzy set theory. If an object a is valued higher than another object b, there exists a finite n such that nb is valued at least as high as a. Archimedean axiom is rejected in value theory [12] because some items are infinitely more valuable than others for agents (like life, well-being, etc.). That is why, value theory prescribes, at most, an interval scale on which subjective values can be measured.

In (expected) utility theory with lotteries and subjective probabilities [6], Archimedean axiom, once again, states the same principle: no consequence is infinitely desirable than another. Savage [13] realizes the non-necessary nature of the Archimedean axiom but cannot dismiss it since without it he cannot come up with a ratio scale representation.

3 Archimedean axiom in Fuzzy Set Theory

In order to qualify the claim that the concept of graded membership is an intuitive and valid representation of fuzziness, we undertake this task within the framework provided by measurement theory [9, 12, 11, 15, 10]. In such a theory one can discuss the representation of a qualitative structure by a numerical structure and the meaningfulness of such a representation. The problem of meaningfulness can be summarized as: "Numerical statements are meaningful insofar as they can be translated, using the mapping conventions, into statements about the original qualitative structure." [8].

In view of measurement theory, we investigate the underlying qualitative structure of fuzzy set theory which turns out to be a well studied mathematical concept: an ordered algebraic structure. This view is in accord with the claim that algebra is a suitable tool to analyze logic, which may be disputed.

In [1, 2], the conditions imposed on the qualitative structure are laid out and critically discussed as to their suitability to the cognition of fuzziness.

We take fuzzy propositions to be of the form:

Mary is more intelligent than John is funny

where "intelligent" and "funny" are the fuzzy terms. This can be formalized in an ordered algebraic structure in the following manner: consider a countable set of agent-adjective pairs, $\mathcal{X} = \{(a, F), (b, G), \cdots\}$ and a binary relation, $\succsim$ on X with the following interpretation:

$$(a, F) \succsim (b, G) \iff a \text{ belongs to } F \text{ at least as much as } b \text{ belongs to } G.$$

We briefly summarize basic definitions and the main representation results for ordered algebraic structures.

Definition 1 *The algebraic structure $\langle A, \oplus \rangle$ where A is a nonempty set and $\oplus$ is a binary operation on A is called a* semigroup *if and only if $\oplus$ is associative (i.e., for all $a, b, c \in A$, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$). If there exists $e \in A$ such that for all $a \in A$, $e \oplus a = e \oplus a = a$ the structure $\langle A, \oplus, e \rangle$, is called a* semigroup with identity e *or a* monoid*. Finally, $\langle A, \oplus, e \rangle$ is a* group *if and only if it is a semigroup with identity e and any element of A has an inverse:for all $a \in A$, there exists $b \in A$ such that $a \oplus b = b \oplus a = e$.*

When the algebraic structure is also endowed with an ordering, $\succsim$, we obtain *ordered algebraic structures*.

Definition 2 *Let A be an empty set, $\succsim$ a binary relation on A and $\oplus$ a binary operation on A. $\langle A, \succsim, \oplus \rangle$ is an* ordered structure *if and only if the following axioms are satisfied:*

(weak ordering) $\succsim$ *is connected and transitive,*

(monotonicity) for all $a, b, c, d \in A$, $a \succsim c$ *and* $b \succsim d$ *imply* $a \oplus b \succsim b \oplus d$.

The asymmetric part ($\succ$) and the symmetric complement ($\sim$) of any relation $\succsim$ are defined as usual: $a \succ b$ if and only if $a \succsim b$ and not $b \succsim a$ and $a \sim b$ if and only if $a \succsim b$ and $b \succsim a$.

Adding more properties to an ordered algebraic structure results in specializations of the concept. In this paper, we only consider ordered semigroups (where the concatenation is associative). These are summarized in the following definition:

Definition 3 *Let* $\mathcal{A} = \langle A, \succsim, \oplus \rangle$ *be an ordered algebraic structure such that* $\langle A, \oplus \rangle$ *is a semigroup. Then* $\mathcal{A}$ *is called an* ordered semigroup. *Furthermore, it is said to be:*

Weakly Associative *(WA)* $a \oplus (b \oplus c) \sim (a \oplus b) \oplus c$.

Solvable *(Sv) iff whenever* $a \succ b$ *then there exists* $c \in A$ *such that* $a \succsim b \oplus c$.

Strongly Monotonic *(SM) iff whenever* $a \succsim b$ *then* $a \oplus c \succsim b \oplus c$ *then* $c \oplus a \succsim c \oplus b$.

Homogeneous *(H) iff whenever* $a \succsim b$ *if and only if* $a \oplus c \succsim b \oplus c$ *if and only if* $c \oplus a \succsim c \oplus b$.

Idempotent *(Ip) iff for all* $a \in A$, $a \oplus a \sim a$.

Bounded *(B) iff there exist* u *and* e *in* A *such that: for all* $a \in A$, $u \succ e$, $u \succsim a$ *and* $a \succsim e$.

Archimedean *(Ar) iff for any* $a, b \in A$ *there exists a positive integer* m *such that* $a^{(m)} \succ b$ *where* $a^{(m)}$ *is recursively defined as* $a^1 = a$, $a^{(m)} = a \oplus a^{(m-1)}$.

Continuous *iff* $\oplus$ *is continuous as a function of two variables, using the order topology on its range and the relative product topology on its domain.*

By a representation of an ordered algebraic structure, we mean a real valued function that maps the ordered algebraic structure, $\langle A, \succsim, \oplus \rangle$ to a numerical structure, $\langle X, \geq, S \rangle$, where X is a subset of $\mathbb{R}$, $\geq$ is the natural ordering of real numbers and $S : X \times X \to X$ is a function. Since we focus on ordered semigroups, in the resulting representation, S is necessarily associative.

The boundary condition, asserts the existence of a minimal and a maximal element in set A. Hence, given the weak ordering and the boundaries, one can replace the set A by the familiar interval notation $[e, u]$

The following lemma demonstrates some of the consequences of axioms imposed on a bounded ordered semigroup [14].

Lemma 1 *Let* $\mathcal{A} = \langle A, \succsim, \oplus \rangle$ *be a bounded ordered semigroup with bounds* e *and* u*.Then* $\mathcal{A}$ *also satisfies the following conditions for all* $a, b \in A$*:*

(i) $a \oplus b \succsim sup(a, b)$,

(ii) $u \oplus a \sim a \oplus u \sim u$,

(iii) $a \oplus a \succsim a$.

In [14, Section 5.3] a *function* defined on a closed real interval $[a, e]$, endowed with the natural ordering, $\geq$, is considered. Here, a more abstract structure is considered but their results carry over to our setting without modification since our relation, $\succsim$, is transitive and connected and hence $\sim$ is an equivalence.

Representation theorems with varying uniqueness characteristics can be given for ordered semigroups. These are summarized in the following:

Theorem 1 *The algebraic structure* $\langle A, \succsim, \oplus \rangle$ *is:*

(i) a bounded ordered semigroup *if and only if there exists* $\gamma : [e, u] \to X \triangleq [\underline{x}, \overline{x}]$ *such that,* $a \succsim b \iff \gamma(a) \geq \gamma(b)$, $\gamma(e) = \underline{x}$, $\gamma(u) = \overline{x}$, *and* $\gamma(a \oplus b) = S(\gamma(a), \gamma(b))$ *where* $X \triangleq [\underline{x}, \overline{x}]$ *is a closed subset of* $\mathbb{R}$ *and* S *is an associative, monotonic function such that* $S : [\underline{x}, \overline{x}] \times [\underline{x}, \overline{x}] \to [\underline{x}, \overline{x}]$ *which has* $\underline{x}$ *as its identity. Furthermore,* γ' *is another representation if and only if there exists a strictly increasing function* $\phi : [\underline{x}, \overline{x}] \to [\underline{x}', \overline{x}']$ *with* $\phi(\underline{x}) = \underline{x}'$ *and* $\phi(\overline{x}) = \overline{x}'$ *and such that for all* $x \in [\underline{x}, \overline{x}]$, $\gamma'(x) = \phi(\gamma(x))$ *(ordinal scale).*

(ii) a bounded idempotent semigroup *if and only if all the conditions in (i) are satisfied and* $S = \max$.

(iii) a continuous Archimedean bounded ordered semigroup *if and only if the conditions of (i) are satisfied with* $X = [0, 1]$, *and there exists a strictly increasing continuous function,* $g : [0, 1] \to \bar{\mathbb{R}}^+ = [0, \infty]$ *with* $g(0) = 0$, *such that for all* $x, y \in [0, 1]$, $S(x, y) = g^{[-1]}(g(x) + g(y))$, *where* $g^{[-1]}$ *is the pseudo-inverse of* g *given by:* $g^{[-1]}(\alpha) = g^{-1}(\min\{\alpha, g(1)\})$. *Furthermore,* g *is unique up to a positive constant (ratio scale).*

(iv) a solvable homogeneous Archimedean strongly monotonic ordered semigroup *if and only if it is isomorphic to a sub semigroup of* $\langle \mathbb{R}^+, \geq, + \rangle$. *Moreover, two such isomorphisms are unique up to a positive constant (ratio scale).*

(v) a solvable Archimedean strongly monotonic ordered semigroup *if and only if it is isomorphic to a sub semigroup,* $\langle [0, 1], \geq, S_W \rangle$ *where* $S_W(x, y) = \min\{x + y, 1\}$ *for all* $x, y \in [0, 1]$ *and two such isomorphisms are necessarily equivalent (absolute scale).*

Figure 2 shows the interdependence of measurement axioms, the corresponding scales, and summarizes Theorem 1.

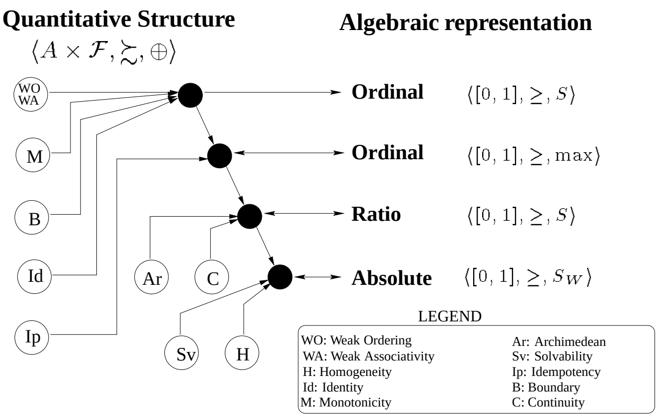

Figure 2. Summary of representations: S is a t-conorm, S_W is the Łukasiewicz co-norm that models disjunction $\oplus$

.

In this formal framework, we discuss the Archimedean axiom. Note that, when one uses continuous, Archimedean triangular norms and conorms as intersection and union in fuzzy set theory, one implicitly commits to the Archimedean axiom.

From Figure 2, one can deduce two things about the role of the Archimedean axiom:

- if you endow the structure with more structure (Homogeneity and Solvability in this case), then the Archimedean axiom is implicitly implied.
- without it you cannot have ratio or absolute scale representations.

But what does the latter exactly mean?

In terms of the above example, the Archimedean axiom in fuzzy set theory implies that there exists a *finite* amount of the quality of being fun, which when attributed to John will make John funnier than Mary is intelligent. We do not allow for two concepts to be infinitesimally different from each other on the basis of their membership to a fuzzy set. Or in other words, if there is a *standard sequence* with which we measure the concept of graded membership, and if it is strictly bounded then it is finite.

It seems that we are in the same boat with value theorists: in order to accept the Archimedean axion we should allow all membership functions to be *comparable* and we should not allow two membership functions to be infinitesimally different from each other. This seems to be acceptable for concepts with linear, continuous universe of discourses like tallness, temperature, etc. However, with multi-dimensional domains (like comfort, humidity, etc.) and some concepts the Archimedean axiom cannot be so easily entertained.

4 Summary

The summary of the discussion is as follows:

- If one commits to any one of the continuous Archimedean t-norms and t-conorms, one also implicitly commits to the Archimedean axiom.
- It is extremely hard to empirically verify the Archimedean axiom in fuzzy set theory.
- However, in many other contexts, the Archimedean axiom is usually accepted for its *normative* appeal rather than its empirical validity. Although this is true for normative theories, if fuzzy set theory is to be applied at all, we believe that its basic axioms need to be validated either empirically or by thought experiments. Archimedean axiom seems to lack both type of validations for fuzzy set theory.
- Without the Archimedean axiom, one cannot attain ratio or absolute scale representations. The resulting representations are *ordinal* in which two truth values can only be compared to each other.
- The functions min and max stand distinctively non-Archimedean. However, their idempotency seems to be a trouble for interactive fuzzy sets.
- One way to dispense with the problematic Archimedean axiom is to consider representations into the field extensions of real numbers (hyper reals). Another way is, of course, to assume that the system is finite.

References

[1] T. Bilgiç. *Measurement-Theoretic Frameworks for Fuzzy Set Theory with Applications to Preference Modelling.* PhD thesis, University of Toronto, Dept. of Industrial Engineering Toronto Ontario M5S 1A4 Canada, 1995.

[2] T. Bilgiç and I. Türkşen. Measurement-theoretic justification of fuzzy set connectives. *Fuzzy Sets and Systems*, 76(3):289–308, December 1995.

[3] T. Bilgiç and I. B. Türkşen. Measurement of membership functions: Theoretical and experimentalwork. To appear in the Handbook of Fuzzy Systems, 1995.

[4] T. Bilgiç and I. B. Türkşen. Measurement-theoretic frameworks for fuzzy set theory. In *Working notes of the IJCAI-95 workshop on "Fuzzy Logic in Artificial Intelligence"*, pages 55–65, August 19, 21 1995. 14th International Joint Conference on Artificial Intelligence Montréal, Canada.

[5] T. L. Fine. *Theories of Probability: An Examination of Foundations*. Academic Press, New York, 1973.

[6] P. C. Fishburn. *Utility theory for decision making*. Operations Research Society of America. Publications in operations research, no. 18. Wiley, New York, 1970.

[7] J. Goguen. L-fuzzy sets. *Journal of mathematical analysis and applications*, 18:145–174, 1967.

[8] D. H. Krantz. From indices to mappings: The representational approach to measurement. In D. R. Brown and J. E. K. Smith, editors, *Frontiers of Mathematical Psychology: Essays in Honor of Clyde Coombs*, Recent Research in Psychology, chapter 1. Springer–Verlag, Berlin, Germany, 1991.

[9] D. H. Krantz, R. D. Luce, P. Suppes, and A. Tversky. *Foundations of Measurement*, volume 1. Academic Press, San Diego, 1971.

[10] R. Luce, D. Krantz, P. Suppes, and A. Tversky. *Foundations of Masurement*, volume 3. Academic Press, San Diego, USA, 1990.

[11] L. Narens. *Abstract Measurement Theory*. MIT Press, Cambridge, Mass., 1986.

[12] F. Roberts. *Measurement Theory*. Addison Wesley Pub. Co., 1979.

[13] L. J. Savage. *The foundations of statistics*. Dover Publications, New York, 2 edition, 1972.

[14] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces*. North-Holland, Amsterdam, 1983.

[15] P. Suppes, D. Krantz, R. Luce, and A. Tversky. *Foundations of Measurement*, volume 2. Academic Press, San Diego, 1989.